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Editorial

Meeting the Fractional Calculus

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When entering in a University, the student that searches for the Exact Sciences area comes across the Calculus subject, right in the first semester. They learn that this discipline will follow them until the end of the course and discover the beauty and the number of possible applications.

In Calculus, the student will learn that the derivative concept, associated with a reason (or an inclination), and the integral concept, associated with an area, merge themselves in the famous Fundamental Theorem of Calculus. When completing this subject, the student will find two others that, at first, may form the base to step into other methodologies, for example, the various integral transforms and their applications.

Therefore, it is worth highlighting, first the differential equations; after all, due to the broad spectrum of applications, it is a game-changer to those who will face a discipline of analysis more targeted to the Mathematics, Applied Mathematics and Physics students, because the others dont have this requirement in their curriculums.

On the other hand, the subject of Complex Variables (or Analytical Functions) provides the student, among others, an elegant manner of recovering results, in particular involving real integrals calculated through the complex plan, as well as the powerful Residue Theorem with its memorable importance in the calculus of inverse transforms.

In general, this material, calculus, differential equations and analytical functions, are covered in the first four semesters, in other words, two of a four or five years of duration course. It is believed that what is coming ahead is part of a possible complement of its formation depending on the particular course choice.

Used to be in the Calculus discipline that multiple questions, elaborated by High School students, are fully understood, for example: What is it for? Where will I use it? Because, in general, after the concept of limit, those questions can be discussed with more property. Its noteworthy that the derivative concept could be introduced in High School, immediately after the concept of Functions, for example, as a growth/degrowth population tax, something palatable for the student of this period.

Lets return to higher education, precisely in the exact sciences courses, where the Calculus discipline makes a presence since the beginning of the course. A frequently asked question: Why isn't there a non-integer order derivative, a half order derivative, for example? Its worth mentioning this kind of questioning by the student because it emerges naturally when discussing Abels integral equation by the Laplaces transform and that, for many researchers, results in the first application of the called arbitrary order calculation, or non-integer order calculation, popularly nicknamed Fractional Calculus.

The problem of nonlocality emerges naturally in the non-integer order calculation. While in the integer-order calculus, as proposed independently by Newton and Leibniz, the derivative is calculated in a point, the non-integer order derivative brings with itself all the past, in particular, we say that it contains the memory term, that is, one nonlocal problem. The nonlocality is a characteristic of non-integer order calculations and, without it; the derivative is just a multiple of an integer-order derivative. Another characteristic of a particular integer-order derivative is the intriguing fact of not showing the null value to the constant derivative. These two factors, the nonlocality and the constant derivative, by themselves, make your study, at least, challenging and, with no doubt, it is the propelling spring for a student to join the "fractional community".

Thus, we'll divide the theme into two research lines, namely: the purely mathematical study where, for example, emerges naturally the study of a class of functions, the so-called Mittag-Leffler functions that arent a particular case of the hypergeometric functions. Those functions are a particular case of the Fox functions, from the Mellin-Barness representations type, where the complex plan performs an important role.

On the other hand, we mention the applications where, in the case of differential non-integer order equations, linear, integral transforms, play an important role, because they reflect the classical manner to approach such equations. It is in this sense that the formulation of endless derivatives begins, among them, beyond the classified as classics, the formulations of Riemann-Liouville, Grünwald-Letnikov and Caputo; those with non-singular core which we mentioned just the first one, the formulation of Caputo-Fabrizio, as well as many others that sometimes change the nucleus, sometimes introduce a special function, until the most general form, the non-integer order derivative of a function about correlating another function, among them, we mention the formulation of the derivatives ψ-Hilfer, recently introduced.

We believe that these two research lines already constituted in enough motivating material to the entering of a student in the "fractional community", but as if that wasn't enough, there are multiple opened problems that by themselves constitute in proper research lines, just to mention: the geometrical interpretation or fractional derivative physics, as well as the study of anomalous diffusion, where the student will come across differential equations, integral equations, among others, linear and nonlinear, in this last one the integral transforms methodology cannot be used.

The year 1695, can be considered the starting point of non-integer order calculus, since, for many researches, a possible exchange of letters between l'Hôpital and Leibniz may have unleashed questions that, from then on, brought carat researchers of Liouville, Fourier, Euler and Riemann, among many others, to focus on the topic, some more, some less, but presenting striking contributions to the study and development of Fractional Calculus.

In 1969, Caputo introduced the integer-order derivative in the integrand and, due to that, the non-integer order of a constant derivative is zero. Therefore, all the formulations in that type, that is the non-integer order integral of an integer-order derivative, attend by the name Caputo-type formulations. Until 1974, the researches in the theme published can be considered sporadic, because the first international conference with a specific theme occurred in the University of New Haven, in the State of Connecticut (EUA). Now, with a research line consolidated, multiple books in the subject started to be published, as well as in the eighties and nineties decades, various researchers around the world made relevant contributions, in particular, related to possible interpretations of fractional derivatives. It is also from this period that the symposiums and congresses are consolidated exclusively to Fractional Calculus, resulting, in the last century, the site FRACALMO (http://www.fracalmo.org/), Fractional Calculus Modeling.

In this century, we can assure that the Fractional Calculus is perfectly consolidated, not only as a research line but also in many centers as graduation and/or postgraduate disciplines, which we understand as an effect of the effort of many researchers. We can ensure that in all big research centers we find professionals working with Fractional Calculus.

Finally, for me, it wasn't different; at the beginning of the nineties decade, I took notice of Professor Mainardi's research, also a graduated physician, by a mail that requested me a preprint of a work. We didnt have the facility of the email, so I responded by sending the preprint, but it didn't have a continuation of, for example, joint research, since, until today, it is of fundamental importance, because it enables the disclosure of the results in a faster and embracing way. When the opportunity arrives, do not let it escape, enjoy it.

In the year 2005, I started dedicating myself to Fractional Calculus, not leaving aside some other collaborations that walked in parallel, until in 2009 occurred the presentation of the first doctorate thesis exclusively about Fractional Calculus. Since then, there were multiple other orientations, both at the doctoral level and in master's theses, culminating with the publishing of many other results in various specialized journals.

In 2015, the first book of Fractional Calculus was published in Portuguese, and in 2019 two others, one of them being a Spanish introduction to Fractional Calculus, both in collaborations, as well as a book of proposed and solved exercises. Decidedly when I asked myself about meeting Fractional Calculus, I was already immersed in the topic and would no longer abandon it.

I conclude by presenting a list of a few references with a brief comment, for those who wish to enter in the topic, and I guarantee you that there is much to be done in such a way that each ones meeting with Fractional Calculus can be, at the very least, gratifying and, why not, unique. I close by thanking the INTERMATHS team, on behalf of its editor and wishing a Happy 2022 to everyone. Success!

References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc., New York, (1993).

This book starts with a great historical survey, prioritizing the researchers that contributed with results associated with fractional calculus. It shows some of the fractional derivative definitions, but unfortunately doesn't mention Caputos name, aiming the fractional differential equations with constant coefficients. Ultimately, it is made a nod to the differential equations with non-constant coefficients.

[2] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Intergrals and Derivatives*, Gordon and Breach Science Publishers, Amsterdam, (1993).

After a brief introduction of historical notes, this book presents a rigorous mathematical treatment of the integrals and the fractional derivatives, aiming applications. It shows a large number of fractional derivatives formulations, but also does not mention Caputo's name, but a Dzherbashyan's version and that many researchers write like Caputo-Dzherbashyan type. This book, a classic nowadays, is intended for students and professionals.

[3] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *The Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, (2006).

This book, also a classic, is intended for students and professionals in the fractional differential equations topic. It displays a series of applications without letting aside the theory. I believe that it must be read before Samko, Kilbas e Marichev book.

[4] J. A. Tenreiro Machado, V. Kiryakova and F. Mainardi, *Recent history of fractional calculus*, Commun. Nonl. Sci. Num. Simulat., **16**, 1140-1153, (2011). https://doi.org/10.1016/j.cnsns.2010.05.027

This article was published in an important specialized journal. It is of easy reading and mentions both other works from the same authors, available in posters, in A3's size, where it is shown, in a timeline, the history of old and new researchers that contributed in a memorable way to fractional calculus.

[5] R. Figueiredo Camargo e E. Capelas de Oliveira, *Cálculo Fracionário*, Editora Livraria da Física, São Paulo, (2015). ISBN 9788578613297

The book presents a brief introduction to fractional calculus. Published in Portuguese, focusing on the fractional derivative formulations in the versions of Riemann-Liouville and Caputo, although others are mentioned, including some that cannot be considered fractional, according to criteria for the possibility of a derivative to be fractional. Differently from the past books, multiple exercises are discussed and some are left to the reader.

[6] J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi and T. Atanacković, *Round Table Discussion - Fractional Calculus: D'où venons nous? Que sommes-nous? Où allons nous?*, Frac. Cal. & Appl. Cal., **19**, 1074-1104, (2016). https://doi.org/10.1515/fca-2016-0059

This is an article worth reading, as it presents various discussions uttered by state-of-the-art researchers. It displays a series of "doors that can be opened" (possibilities), after all, it consists of open problems.

[7] D. S. Oliveira and E. Capelas de Oliveira, *On the generalized* (k,ρ)-*fractional derivative*, Progr. Fract. Differ. Appl., **2**, 133-145, (2018). http://dx.doi.org/10.18576/pfda/040207

In this article, a natural consequence coming from the authors doctorate thesis introduces a new derivative that contains many other particular cases. This formulation contemplates a new class of fractional derivatives, those which involve a new parameter, beyond the respective derivative order.

[8] J. V. da Costa Sousa and E. Capelas de Oliveira, *On the* ψ*-Hilfer fractional derivative*, Commun. Nonl. Sci. Numer. Simulat., **60**, 72-91, (2018). https://doi.org/10.1016/j.cnsns.2018.01.005

In this article, a natural consequence coming from the authors doctorate degree thesis introduces a new derivative that contains many others as particular cases. This formulation contemplates a new class of fractional derivatives, those which have the fractional derivative related to another function and contemplate a broad class of fractional derivatives obtained as particular cases either from the parameter, or from the function.

[9] G. Sales Teodoro, J. A. Tenreiro Machado, and E. Capelas de Oliveira, *A review of definitions of fractional derivatives and other operators*, J. Comput. Phys., **388**, 195-209, (2019). https://doi.org/10.1016/j.jcp.2019.03.008

The article, a natural continuation of another authors article, is a consequence from the author's doctorate degree thesis, collects and subdivides multiple fractional derivative formulations, highlighting the considered classical formulations, as well as shows the validation criteria to all formulations covered there.

[10] A. R. Gómez Plata y E. Capelas de Oliveira, *Introducción al Cálculo Fraccional*, Editorial Neogranadina, Bogotá, (2019). https://doi.org/10.18359/9789588795812

This Spanish book shows in simple ways the first notions of fractional calculus, discussing only the fractional derivatives formulations according to Riemann-Liouville and Caputo, and counts with various applications where the Mittag-Leffler's function plays a fundamental role. The text does not come with proposed exercises for the reader.

[11] E. Capelas de Oliveira, *Solved Exercises in Fractional Calculus*, Studies in Systems, Decision and Control, vol. 240, Springer Nature Switzerland AG, (2019). https://doi.org/10.1007/978-3-030-20524-9

Until this date, it was the only book of solved and proposed exercises, with solutions, where the student and/or professional can go through the fractional calculus study in a relatively natural order. After a brief historical survey, their special functions are displayed, highlighting Mittag-Leffler's function, integral transforms, Laplace Fourier and Mellin, and the various formulations, highlighting those considered classic. At the end, it offers a chapter all dedicated to applications regarding multiple articles published in specialized journals and those considered classics.

[12] E. Capelas de Oliveira, S. Jarosz, and J. Vaz Jr., *On the mistake in defining fractional derivative using a non-singular kernel*, arXiv:1912.04422v3.

The authors question and prove that some formulations cannot be considered fractional. To do this, they use the methodology of Laplace's transform.

[13] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monograph in Mathematics, Second Edition, Heidelberg, (2021). https://doi.org/10.1007/978-3-662-43930-2

It surely is an interesting compendium about the Mittag-Leffler function, namely by one of the authors as the queen of the special functions in fractional calculus. There are presented and discussed the Mittag-Leffler functions according to a number of parameters, from the classical, as introduced by Mittag-Leffler, containing a parameter, until the generalized containing an integer number of parameters. There are various applications, going through deterministic and stochastic models. Six excellent appendices conclude the text. A series of exercises presented by the chapters is left in charge of the reader.

[14] J. Vaz Jr. and E. Capelas de Oliveira, *On the fractional Kelvin-Voigt oscillator*, Math. Eng., 4, 1-23, (2022). https://doi.org/10.3934/mine.2022006

Here, differently from the majority of the authors, it discuss the model that attends by the name of fractional oscillator, but having the fractional derivative not in the second order derivative, but associated with the first order, recovered from a convenient limit process.

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Breve Biografia

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Physics doctorate at the State University of Campinas (Unicamp). He made the postdoctoral degree with Università di Perugia, Italy. He is currently a professor holder in the Applied Mathematics Department by the Institute of Mathematics and Statistics and Scientific Computing of UNICAMP. He has experience in the Physics area, with emphasis on Mathematical Physics Methods, acting mainly in the topics: fractional Integro-differential calculus, special functions, analytical functions and differential equations.