

# Fifth-order $A(\alpha)$ -stable block hybrid adams-moulton method for solutions of predator-prey and lorenz systems

O. Adedire <sup>a,b</sup> and P.C. Mordi <sup>c</sup>

<sup>a</sup>Department of Mathematics, University of Jos, Plateau State, Nigeria; <sup>b</sup>Department of Statistics, Federal College of Forestry, Jos, Plateau State, Nigeria; <sup>c</sup>Department of Mathematics and Statistics, University of Windsor, Ontario Canada.

\* Correspondence: [dharens@gmail.com](mailto:dharens@gmail.com)

**Abstract:** Problems associated with nonlinearity in predator-prey and the chaotic nature embedded in the Lorenz system place a significant challenge on numerical methods for their solutions. Some numerical methods may become unstable as the step size increases. In this study, a fifth-order  $A(\alpha)$ -stable ( $\alpha = 89.9^\circ$ )  $k$ -step block hybrid Adams-Moulton method (BHMM) was derived incorporating  $\frac{16}{9}$  as an off-step interpolation point using multistep collocation and matrix inversion techniques. The choice of the off-step point of the BHMM was in the upper part of the interval of interpolation points. It was shown that the derived block method was consistent and zero-stable, hence a convergent block method. Numerical simulations of predator-prey and Lorenz systems with the newly derived  $k = 3$  BHMM indicated that it was adequate and compared well with Matlab ode23s.

**Keywords:** hybrid; interpolation; multistep; collocation; matrix-inversion.

**Classification MSC:** Primary 65C20; Secondary 65C20; 65L07; 65L20; 65Y20; 74S30.

## 1 Introduction

Mathematical models have a tendency to predict species interactions and atmospheric convection. Two of such models are predator-prey and Lorenz systems. While predator-prey interactions emerged from the study of ecology, the Lorenz system proposed by [13] came from the study of atmospheric convection. Some studies in the literature have considered predator-prey dynamics using ordinary differential equations (ODEs). Several other types of models have been derived - in extended form - to investigate species interactions, such as the Lotka-Volterra, Rosenzweig-MacArthur and Holling-Tanner models. The predator-prey model considered in this study is of the form:

$$\begin{aligned}\frac{du}{dt} &= \alpha u - \beta uv \\ \frac{dv}{dt} &= \delta uv - \gamma v\end{aligned}\tag{1.1}$$

with variables  $u$  and  $v$  showing both prey and predator populations as well as parameters  $\alpha, \beta, \delta$  and  $\gamma$  representing how the variables interact. For further details on the predator-prey system, readers may check [6, 11, 16].

Other studies in the literature have considered the Lorenz system, which consists of a system of three coupled ODEs with a tendency to show convection within a fluidized layer and exhibit sensitive dependence on initial conditions. It was noted that little changes in the starting values of the Lorenz system can produce drastically different trajectories. The Lorenz system considered in this study is of the form:

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x + \alpha y \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz\end{aligned}\tag{1.2}$$

where  $\alpha$ ,  $r$  and  $b$  are constants defined in [13]. Further details on the Lorenz system can be found in [1–3, 14]. However, nonlinearity and stiffness embedded in predator-prey and Lorenz systems place a significant challenge on numerical methods for their solutions. Various numerical methods have been proposed such as Runge-Kutta methods and multistep implicit methods among others. Runge-Kutta methods have wide range of applications but they can significantly accumulate errors due to their exponential sensitivity to chaos. Multistep methods have the advantage of improving stability in chaos, but some of their classes have limited capability for handling large temporal steps usually required for long-term dynamics. Implicit methods possess good stability but with increased costly computational complexity [1, 4, 5, 17, 20, 22]. Several forms of numerical methods have been used in the literature to solve nonlinear and stiff problems such as predator-prey and Lorenz systems; for further research on block numerical methods and other integrators, readers can consult [10, 12, 18, 19, 21].

In this study, we focus on block hybrid form of Adams-Moulton method because the classical Adams-Moulton method has some drawbacks concerning stability. Motivation to embark on the research centres on deriving an implicit numerical method with numerical stability for moderate time steps crucial for capturing long term dynamics between species in predator-prey and Lorenz systems while offering accuracy and computational efficiency which can be obtained from block methods and at the same time capable of minimizing function evaluations thereby leading to computationally efficient solutions. The block hybrid Adams-Moulton method - due to its hybrid structure - possess captivating capability by leveraging past solution values to solve challenging ODEs from different models including predator-prey and Lorenz systems with adequate stability, efficiency and accuracy thereby offering advantages over single-step methods.

The aim of this study is to develop a fifth-order  $A(\alpha)$ -stable  $k$ -step block hybrid Adams-Moulton method for solutions of predator-prey and Lorenz systems. This will be achieved with incorporation of an off-step interpolation point. The specific objectives include analysis of the block method and its numerical test on the models. The remaining part of the paper is arranged as follows: section 2 considers derivation of the block method, section 3 investigates convergence analysis. Section 4 handles stability analysis and plots of the region of absolute stability, section 5 considers numerical simulations of the models and conclusion comes up in section 6.

## 2 Derivation of the block method

Block integrator which shall be considered in this study will be derived from some continuous  $k$ -step linear multistep methods whose coefficients are assumed polynomials

obtained by the use of multistep collocation [15] for the solution of the system of first order ordinary differential equations. To this end, consider a polynomial

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x)y(x_{n+j}) + h \sum_{j=0}^{m-1} \beta_j(x)f(\bar{x}_j, y(\bar{x}_j)), \quad x_n \leq x \leq x_{n+k} \quad (2.1)$$

of degree  $p = t + m - 1, t > 0, m > 0$ , where  $\alpha_j(x)$  and  $\beta_j(x)$  are continuous coefficients defined by

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1}x^i; j \in \{0, \dots, t-1\} \quad (2.2)$$

$$h\beta_j(x) = h \sum_{i=0}^{t+m-1} \beta_{j,i+1}x^i; j \in \{0, \dots, m\} \quad (2.3)$$

where  $x_{n+j}$  in (2.1) are interpolation points  $t(0 < t \leq k)$  arbitrarily chosen from  $\{x_n, x_{n+1}, \dots, x_{n+k-1}\}$  and  $m$  are collocation points.

The coefficients  $\alpha_j(x)$  and  $\beta_j(x)$  of (2.1) defined by (2.2) and (2.3) can be obtained from columns of matrix  $C$  defined by

$$C = D^{-1} \quad (2.4)$$

where  $D$  is multistep collocation matrix defined by

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdot & \cdot & \cdot & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdot & \cdot & \cdot & x_{n+1}^{t+m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdot & \cdot & \cdot & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdot & \cdot & \cdot & (t+m-1)\bar{x}_0^{t+m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 2\bar{x}_{m-1} & \cdot & \cdot & \cdot & (t+m-1)\bar{x}_{m-1}^{t+m-1} \end{bmatrix} \quad (2.5)$$

and matrix  $D$  has dimension  $(t+m) \times (t+m)$ , using matrix operations on (2.4) gives  $DC = I$  where  $I$  is an identity matrix of dimension  $(t+m)$ . For more elaborate description of equations (2.1)-(2.5), readers should see [15]. We shall consider continuous formulation of hybrid  $k$ -step Adams-Moulton method subsequently.

## 2.1 Continuous formulation of three-step Adams-Moulton method incorporating one off-grid interpolation point

Let  $k = 3, t = 2, m = 4$  and let  $x = x_{n+\frac{16}{9}}$  be off-grid interpolation, then (2.1) becomes

$$y(x) = \alpha_1(x)y_{n+1} + \alpha_{\frac{16}{9}}(x)y_{n+\frac{16}{9}} + h [\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}] \quad (2.6)$$

and (2.5) gives

$$D = \begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+\frac{16}{9}} & x_{n+\frac{16}{9}}^2 & x_{n+\frac{16}{9}}^3 & x_{n+\frac{16}{9}}^4 & x_{n+\frac{16}{9}}^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \end{pmatrix}. \quad (2.7)$$

Using Maple software for  $C = D^{-1}$  and substitution of the continuous coefficient into (2.6) gives the following continuous formulation of the three-step hybrid Adams-Moulton method as

$$y(x) = \left( \begin{aligned} & -\frac{183808}{190169h^5} \left( h^4 + \left( \frac{9x}{16} - \frac{9x_n}{16} \right) h^3 - \frac{53541(x-x_n)^2 h^2}{5744} \right. \\ & + \frac{75087(x-x_n)^3 h}{11488} - \frac{6561(x-x_n)^4}{5744} \left. \right) \left( \frac{9x_n}{16} + h - \frac{9x}{16} \right) y_{n+1} \\ & + \left( \frac{373977(h-x+x_n)^2}{190169h^5} \left( h^3 + (2x - 2x_n) h^2 - \frac{33(x-x_n)^2 h}{19} + \frac{6(x-x_n)^3}{19} \right) y_{n+\frac{16}{9}} \right. \\ & + \left( -\frac{1168(h-x+x_n)^2}{3881h^4} \left( h^2 + \left( -\frac{111x}{146} + \frac{111x_n}{146} \right) h + \frac{121(x-x_n)^2}{876} \right) \left( \frac{9x_n}{16} + h - \frac{9x}{16} \right) \right) f_n \\ & + \left( -\frac{147136h-147136x+147136x_n}{81501h^4} \left( h^3 + \left( \frac{25x}{16} - \frac{25x_n}{16} \right) h^2 - \frac{29641(x-x_n)^2 h}{18392} \right. \right. \\ & + \frac{11595(x-x_n)^3}{36784} \left. \left. \left( \frac{9x_n}{16} + h - \frac{9x}{16} \right) \right) f_{n+1} + \left( -\frac{5248(h-x+x_n)^2}{11643h^4} \left( h^2 + \left( \frac{41x}{16} - \frac{41x_n}{16} \right) h \right. \right. \right. \\ & - \frac{1077(x-x_n)^2}{1312} \left. \left. \left( \frac{9x_n}{16} + h - \frac{9x}{16} \right) \right) f_{n+2} + \left( \frac{320(h-x+x_n)^2}{11643h^4} \left( h^2 + \left( \frac{41x}{16} - \frac{41x_n}{16} \right) h \right. \right. \right. \\ & \left. \left. \left. - \frac{113(x-x_n)^2}{80} \right) \left( \frac{9x_n}{16} + h - \frac{9x}{16} \right) \right) f_{n+3} \right) \end{aligned} \right). \quad (2.8)$$

Evaluating (2.8) at  $x = x_n$ ,  $x = x_{n+2}$  and  $x = x_{n+3}$  and its derivatives at  $x = x_{n+\frac{16}{9}}$  gives the following discrete schemes;

$$\left. \begin{aligned} y_{n+1} &= -\frac{190169}{183808} y_n + \frac{373977}{183808} y_{n+\frac{16}{9}} - \frac{3577}{11488} h f_n - \frac{16093}{8616} h f_{n+1} - \frac{2009}{4308} h f_{n+2} \\ &+ \frac{245}{8616} h f_{n+3} \\ y_{n+\frac{16}{9}} &= y_{n+1} - \frac{46991}{6298560} h f_n + \frac{80941}{262440} h f_{n+1} + \frac{27167}{31680} h f_{n+\frac{16}{9}} - \frac{51107}{131220} h f_{n+2} \\ &+ \frac{75803}{8660520} h f_{n+3} \\ y_{n+2} &= -\frac{26344}{190169} y_{n+1} + \frac{216513}{190169} y_{n+\frac{16}{9}} + \frac{14}{11643} h f_n - \frac{3683}{81501} h f_{n+1} + \frac{1864}{11643} h f_{n+2} \\ &- \frac{19}{11643} h f_{n+3} \\ y_{n+3} &= \frac{347633}{190169} y_{n+1} - \frac{157464}{190169} y_{n+\frac{16}{9}} - \frac{121}{3881} h f_n + \frac{61952}{81501} h f_{n+1} + \frac{18755}{11643} h f_{n+2} \\ &+ \frac{3542}{11643} h f_{n+3} \end{aligned} \right\}. \quad (2.9)$$

Equation (2.9) is the block hybrid Adams-Moulton method for this study, its convergence analysis and region of absolute stability will be considered subsequently.

### 3 Convergence analysis of the block method (2.9)

In this section, convergence analysis of the derived block method (2.9) will be investigated. We first check zero stability, hence reformulate (2.9) using approach found in Fatunla [8] as;

$$M^{(1)} Y_{n+i} = M^{(0)} Y_{n-i} + h(B^{(1)} F_{n+i} + B^{(0)} F_{n-i}), \quad i \in \mathbb{R} \quad (3.1)$$

where

$$M^{(1)} = \begin{pmatrix} 1 & -\frac{373977}{183808} & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{26344}{190169} & -\frac{216513}{190169} & 1 & 0 \\ -\frac{347633}{190169} & \frac{157464}{190169} & 0 & 1 \end{pmatrix}, M^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{190169}{183808} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.2)$$

and

$$B^{(1)} = \begin{pmatrix} -\frac{16093}{8616} & 0 & -\frac{2009}{4308} & \frac{245}{8616} \\ \frac{80941}{262440} & \frac{27167}{31680} & -\frac{51107}{131220} & \frac{75803}{8660520} \\ -\frac{3683}{81501} & 0 & \frac{11643}{18755} & -\frac{11643}{3542} \\ \frac{61952}{81501} & 0 & \frac{11643}{11643} & \frac{11643}{11643} \end{pmatrix}, B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{3577}{11488} \\ 0 & 0 & 0 & -\frac{46991}{6298560} \\ 0 & 0 & 0 & \frac{14}{11643} \\ 0 & 0 & 0 & -\frac{121}{3881} \end{pmatrix}. \quad (3.3)$$

The first characteristics polynomial is given as,

$$\begin{aligned} \rho(\lambda) &= \det(\lambda M^{(1)} - M^{(0)}) \\ &= |\lambda M^{(1)} - M^{(0)}| \\ &= 0. \end{aligned} \quad (3.4)$$

Thus, this leads to,

$$\begin{aligned} \rho(\lambda) &= \left| \lambda \begin{pmatrix} 1 & -\frac{373977}{183808} & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{26344}{190169} & -\frac{216513}{190169} & 1 & 0 \\ -\frac{347633}{190169} & \frac{157464}{190169} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -\frac{190169}{183808} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \\ &= \frac{190169}{183808} \lambda^4 - \frac{190169}{183808} \lambda^3 \\ &\Rightarrow \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0. \end{aligned} \quad (3.5)$$

Since  $|\lambda_i| \leq 1, i = 1, 2, 3, 4$ , then by [7], the block method (2.9) is zero stable and has schemes of order  $p = 5$  with error constants  $-\frac{76489}{13957920}, -\frac{671251}{765275040}, \frac{2827}{18861660}$ , and  $-\frac{106843}{18861660}$  respectively. Since the block method (2.9) has order  $p > 1$  and is zero stable, then it is convergent by [9].

## 4 Region of absolute stability of BHAMM (2.9)

Root boundary locus method is used to obtain absolute stability region of the method (2.9). The BHAMM (2.9) is said to have a region of absolute stability  $R_{As}$ , if it is absolutely stable for all  $\hat{h} \in R_{As}$  where  $R_{As}$  is a region of complex  $\hat{h}$  - plane,  $z = h\lambda$  hence we obtain the stability polynomial;

$$\begin{aligned} \pi(r, z) &= -\frac{190169r^4}{183808} + \frac{190169r^3}{183808} + \frac{2091859r^3z^4}{33085440} + \frac{1331183r^4z}{827136} - \frac{2472197r^4z^2}{2205696} + \frac{4373887r^4z^3}{9925632} \\ &\quad - \frac{190169r^4z^4}{2067840} + \frac{2472197r^3z}{1654272} + \frac{2091859r^3z^2}{2205696} + \frac{6655915r^3z^3}{19851264} \end{aligned}$$

whose plot is shown in Figure 1.

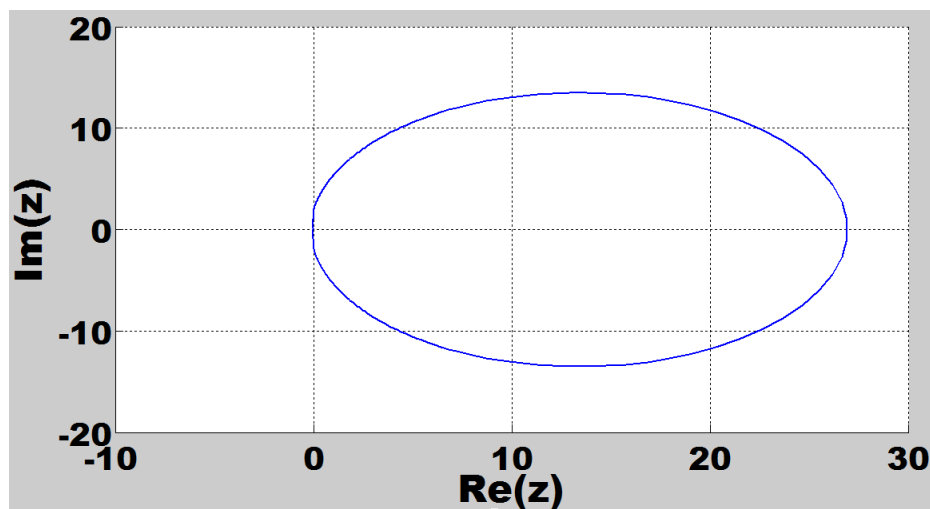


Fig. 1. Region of absolute stability of (2.9)

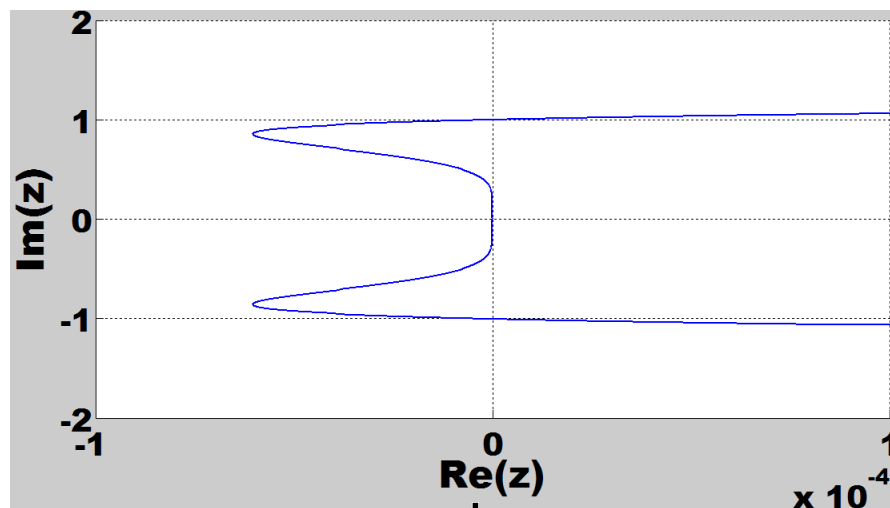
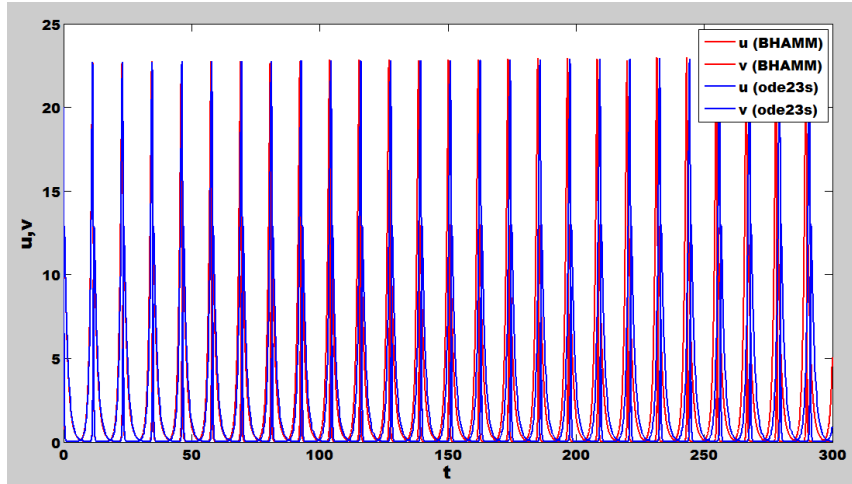


Fig. 2. Detail near the origin of the region of absolute stability of (2.9).

Observe that Figure 1 appeared as if the block method (2.9) is A-stable but detailed outlook of the same figure when rescaled as shown in Figure 2 indicated that it is an  $A(\alpha)$ -stable method. However, the scale of enlargement is very large which is why the pointed part in Figure 2 is conspicuous. Further calculations on the extent of its angle  $\alpha$  give  $\alpha = 89.99^\circ$ . The extent of the size of the angle is the reason why Figure 1 appeared as if the block method is A-stable. Therefore the block method (2.9) is  $A(\alpha)$ -stable.

## 5 Numerical simulation

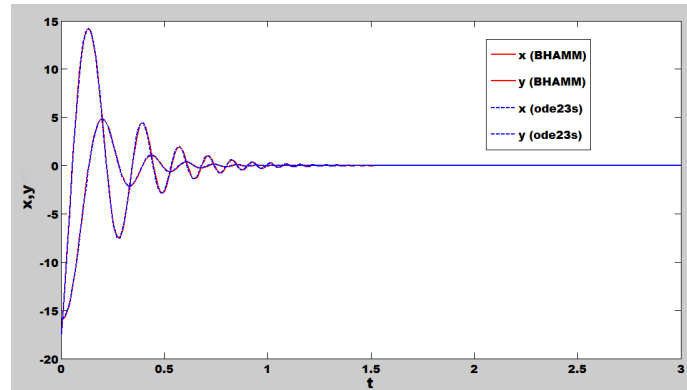
The solutions of (1.1) using parameters  $\alpha = 1.2, \beta = 0.6, \delta = 0.3, \gamma = 0.8, h = 0.1$  for initial conditions  $u(0) = 20$  and  $v(0) = 5$  are shown in Figure 3.



**Fig. 3.** Comparison of BHAMM (2.9) with Matlab ode23s for parameters  $\alpha = 1.2, \beta = 0.6, \delta = 0.3, \gamma = 0.8, h = 0.1$  for solutions of (1.1).

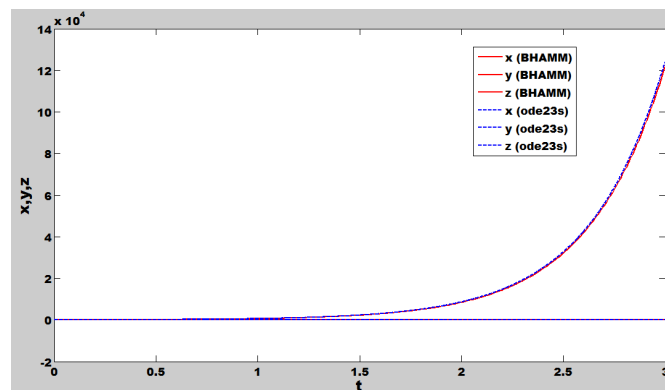
It can be seen from Figure 3 that the newly derived BHAMM (2.9) compete favourably close to the inbuilt Matlab ode23s designed for stiff problems. The solutions shown in the Figure 3 indicate further that the block method is capable of handling nonlinear ODEs. Though not shown here, the new block method possess ability to handle nonstiff problems of ODEs as well.

Analogously, the solutions to (1.2) using parameters  $\alpha = 10, r = 23.5, b = \frac{8}{3}, x(0) = -15.8, y(0) = -17.48, z(0) = 35.64, h = 0.01$  are shown in Figure 4 for solutions of  $x$  and  $y$  only.



**Fig. 4.** Solutions of (1.2) using BHAMM (2.9) is compared with Matlab ode23s for parameters  $\alpha = 10, r = 23.5, b = \frac{8}{3}, h = 0.01$  for  $x$  and  $y$  variables only.

Observe also that for Figure 4, variables  $x$  and  $y$  only were shown in the dynamics of the Lorenz system to enable us view oscillatory nature of  $x$  and  $y$  adequately. It can be seen that the BHAMM (2.9) showed solutions which are similar to those generated by the inbuilt Matlab ode23s. Also for all the three variables  $x, y$  and  $z$ , solutions to (1.2) are shown in Figure 5 using same parameters  $\alpha = 10, r = 23.5, b = \frac{8}{3}, x(0) = -15.8, y(0) = -17.48, z(0) = 35.64, h = 0.01$ .



**Fig. 5.** Simulation of (1.2) for variables  $x, y$  and  $z$  using BHAMM (2.9) compared with Matlab ode23s for parameters  $\alpha = 10, r = 23.5, b = \frac{8}{3}, h = 0.01..$

From the results also presented in Figure 5, the in-built Matlab ode23s produced similar results to those results simulated from BHAMM (2.9).

## 6 Conclusion

In this research, a fifth-order  $A(\alpha)$ -stable  $k$ -step block hybrid Adams-Moulton method was developed for solutions of predator-prey and Lorenz systems. The feasibility of the block hybrid method which incorporated  $\frac{16}{9}$  as an off-grid interpolation point was based on multistep collocation with the use of the matrix inversion technique. This approach modified the classical Adams-Moulton method, thereby improving its very limited stability property for numerical simulation of nonlinear and chaotic systems. Results obtained from use of the newly derived block hybrid method were compared to those obtained from in-built Matlab ode23s. It was shown that the fifth-order, three-step block hybrid method competes favorably with in-built Matlab ode23s for nonlinear and chaotic systems.

**Conflicts of Interest:** The authors declare no conflict of interest in the writing of the manuscript, or in the decision to publish the results.

## ORCID

O. Adedire  <https://orcid.org/0000-0003-3790-7776>

P.C. Mordi  <https://orcid.org/0009-0006-3265-4449>

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