
The Classical Particle Coupled to External Electromagnetic Field – Symmetries and Conserved Quantities

G. D. Barbosa **R. Thibes,**

Universidade Estadual do Sudoeste da Bahia

Departamento de Estudos Básicos e Instrumentais

E-mail:gabrielbarbosa2@hotmail.com thibes@uesb.edu.br,

R. R. Ferreira

Universidade Estadual do Sudoeste da Bahia

Departamento de Ciências Exatas

E-mail:renataferreirad@hotmail.com

Resumo

Consideramos uma partícula clássica acoplada a um campo eletromagnético externo, em ambos os regimes relativístico e não-relativístico. O acoplamento é construído através do potencial eletromagnético satisfazendo as equações de Maxwell clássicas. Revemos o teorema de Noether a nível clássico associando simetrias infinitesimais a quantidades conservadas. As simetrias fundamentais do espaço-tempo são investigadas considerando uma ação não-relativística e outra relativística em um referencial particular. Considerando algumas hipóteses de simetria sobre a variação da ação calculamos em detalhes as quantidades conservadas.

Palavras-chave Simetrias, teorema de Noether, ação clássica, constantes de movimento, grupo de Lorentz.

Abstract

We consider a classical particle minimally coupled to an external electromagnetic field, in both non-relativistic and relativistic regimes. The coupling is constructed via the electromagnetic potential which is assumed to satisfy the classical Maxwell equations. We review Noether's theorem at classical level associating infinitesimal symmetries to conserved quantities. The fundamental space-time symmetries are investigated considering a non-relativistic and a relativistic action in a particular reference frame. Considering some symmetry hypothesis on the action variation we work out in detail the corresponding conserved quantities.

Keywords Symmetries, Noether's theorem, classical action, conserved quantities, Lorentz group.

1 Introduction

The celebrated Noether's theorem [1, 2, 3], originally stated and proved by the German mathematician Emmy Noether in 1915 and first published in 1918 [1], relates continuous differentiable symmetries of the classical action to conserved quantities, i.e., constants of motion. Such connection is achieved through Hamilton's principle of least action [2, 3, 4].

That is a central result in theoretical physics, with deep consequences not only to Classical Physics but also to Quantum Field Theory, for example in the study of Noether currents and

anomalies at quantum level [5]. The latter ones, for instance, concern classically conserved quantities which are not quantum mechanically conserved; i.e., anomalies are related to the breaking of Noether's theorem at quantum level [5, 6].

Besides its practical applications, Noether's theorem has also a certain philosophical appeal. It relates the more fundamental conservation laws of nature, such as energy and momentum, to simple symmetries of the very spacetime continuum we live in, such as time and space translations. It also plays an important role in relating the invariance of physical laws under symmetry operations, including for instance the equivalence of different inertial frames, with corresponding conserved quantities [2, 3].

The present paper is organized as follows. In section 2 below, we discuss symmetries of the action in classical mechanics. We introduce and prove Noether's theorem using the Lagrangian formalism. In section 3 we discuss the nonrelativistic particle coupled to external electromagnetic field, applying Noether's theorem and calculating the conserved quantities. In section 4 we generalize the previous results to the relativistic particle, particularly considering Lorentz symmetries. We close in the last section with some perspectives and concluding remarks.

2 Noether's Theorem

Consider a classical system described by n generalized coordinates q_i , $i = 1, \dots, n$, with time evolution governed by the Lagrangian function $L = L(q_i, \dot{q}_i, t)$. The action S of the system is given by the integral

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t) dt. \quad (1)$$

The real parameter t represents the physical time.

Loosely speaking, Noether's theorem states that to each continuous symmetry of action (1) corresponds a constant of motion. Suppose we perform an infinitesimal transformation in the coordinates $q_i(t)$ and time t ; if the action (1) remains invariant then there is a conserved quantity for each real parameter of the transformation. We shall be more explicit in this statement and proof below.

Hamilton's principle of least action applied to (1) leads to the Euler-Lagrange equations of motion [2, 3]

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (2)$$

whose solution $q_i = q_i(t)$ gives us the temporal evolution of the system. Associated to the system (1) we define the energy function h as

$$h(q_i, \dot{q}_i, t) = \sum_{i=1}^n \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L, \quad (3)$$

also known as Jacobi integral. A simple calculation, using (2), leads to

$$\frac{dh}{dt} = - \frac{\partial L}{\partial t}. \quad (4)$$

Thus we see that if the Lagrangian of the system does not depend explicitly on time, then the Jacobi integral (3) is a constant of motion.

Consider now the q_i and t coordinates transformation

$$\begin{aligned} t &\rightarrow t' = t + \epsilon X(q, t), \\ q_i &\rightarrow q'_i = q_i + \epsilon \Psi_i(q, t), \end{aligned} \quad (5)$$

where ϵ represents an infinitesimal real parameter and X and Ψ_i are given real functions of the generalized coordinates. The corresponding variation of the action (1) reads

$$\Delta S = \int_{t'_1}^{t'_2} L(q'_i(t'), \frac{dq'_i(t')}{dt'}, t') dt' - \int_{t_1}^{t_2} L(q_i(t), \frac{dq_i(t)}{dt}, t) dt. \quad (6)$$

At this point we may finally state and prove Noether's theorem:

Theorem (Noether)

Given a classical physical system governed by action S in (1), and an infinitesimal transformation of variables (5), if the corresponding variation of the action (6) assumes the form $\Delta S = \epsilon G(q(t), t)|_{t_1}^{t_2}$ for some (possibly zero) function $G(q, t)$, then

$$C \equiv \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} (\dot{q}_i X - \Psi_i) - LX + G \quad (7)$$

is a constant of motion. In other words, if the transformation (5) is a symmetry of the Lagrangian up to a total derivative then

$$\frac{dC}{dt} = 0, \quad (8)$$

with C given by (7).

proof: Applying the infinitesimal transformation (5) to (1) we may rewrite the action variation (6) as

$$\begin{aligned} \Delta S &= \int_{t_1}^{t_2} L(q_i + \epsilon \Psi_i, \dot{q}_i + \epsilon \xi_i, t + \epsilon X)(1 + \epsilon \dot{X}) dt - \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt, \\ &= \int_{t_1}^{t_2} \left\{ L(q_i, \dot{q}_i, t) + \sum_{i=1}^n \left[\frac{\partial L}{\partial q_i} \epsilon \Psi_i + \frac{\partial L}{\partial \dot{q}_i} \epsilon \xi_i \right] + \frac{\partial L}{\partial t} \epsilon X \right\} (1 + \epsilon \dot{X}) dt - \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt, \end{aligned}$$

where we have defined

$$\xi_i \equiv \dot{\Psi}_i - \dot{q}_i \dot{X}. \quad (9)$$

Reagrouping and keeping only terms up to first order in ϵ we obtain

$$\Delta S = \epsilon \int_{t_1}^{t_2} \left\{ \sum_{i=1}^n \left[\frac{\partial L}{\partial q_i} \Psi_i + \frac{\partial L}{\partial \dot{q}_i} \xi_i \right] + \frac{\partial L}{\partial t} X + L \dot{X} \right\} dt. \quad (10)$$

Now the $\Delta S = \epsilon G(q(t), t)|_{t_1}^{t_2}$ hypothesis leads immediately to Noether's condition

$$\sum_{i=1}^n \left[\Psi_i \frac{\partial L}{\partial q_i} + (\dot{\Psi}_i - \dot{q}_i \dot{X}) \frac{\partial L}{\partial \dot{q}_i} \right] + L \dot{X} + \frac{\partial L}{\partial t} X = \dot{G}. \quad (11)$$

Finally using the Euler-Lagrange equations (2) and relations (3) and (4), Noether's condition (11) can be rewritten as

$$\sum_{i=1}^n \left[\Psi_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \dot{\Psi}_i \frac{\partial L}{\partial \dot{q}_i} \right] - h \dot{X} - \frac{dh}{dt} X = \frac{d}{dt} \sum_{i=1}^n \left\{ \frac{\partial L}{\partial \dot{q}_i} \Psi_i - h X \right\} = \dot{G}, \quad (12)$$

or simply

$$\frac{d}{dt} \left[\sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} (\dot{q}_i X - \Psi_i) - LX + G \right] = 0, \quad (13)$$

concluding thus the proof of the theorem.

When the function $G(q, t)$ identically vanishes, the transformation is a full symmetry of the Lagrangian, otherwise the transformed Lagrangian differs from the original one by the total time derivative $\epsilon \dot{G}(q, t)$ generating thus the same equations of motion. Naturally, for a given action S , it is possible to have more than one conserved quantity C ; for each independent symmetry (5) of the same action we have a different conserved quantity (7).

3 Non Relativistic Point Particle in External Electromagnetic Field

Let us consider a classical nonrelativistic particle embedded in an external electromagnetic field. The particle has mass m and electrical charge q and is described by the position vector $\mathbf{r}(t)$ with respect to a inertial frame. The Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}, t)$ of this system can be constructed from the particle kinetic energy minus the interacting potential energy between the particle and the electromagnetic field. In Gaussian units, we may thus write it as

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{m\dot{\mathbf{r}}^2}{2} - q\varphi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t), \quad (14)$$

where $\varphi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ represent respectively the scalar and vector potentials, and c is the vacuum speed of light. We are tacitly assuming ordinary three-dimensional space.

The potentials $\varphi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are external fields in the sense that they represent given functions of space-time. Different functions φ and $\mathbf{A}$ lead to distinct electromagnetic field configurations $\mathbf{E}$ and $\mathbf{B}$ which are given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}. \end{aligned} \quad (15)$$

The Euler-Lagrange equations derived from (14) read

$$m\ddot{\mathbf{r}} + \frac{q}{c} \dot{\mathbf{A}} = -q\nabla \left(\varphi - \frac{\dot{\mathbf{r}}}{c} \cdot \mathbf{A} \right), \quad (16)$$

and can be rewritten using (15) as

$$m\ddot{\mathbf{r}} = q\mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B}, \quad (17)$$

which can be promptly recognized as Newton's second law for a particle moving under the Lorentz electromagnetic force. Summarizing, the Lagrangian (14) consistently describes a non-relativistic particle coupled to a given external electromagnetic field, with equations of motion (17). Now we turn our attention to the symmetries and conserved quantities of (14).

What are the symmetries of the action associated to Lagrangian (14)? The answer obviously depends on the functional form of the potentials φ and $\mathbf{A}$. Let us analyse in the following the possibility of realization of the fundamental space-time symmetries in (14), namely, time and spatial translations, spatial rotations and boosts.

Time and Spatial Translations

The energy function (3) corresponding to Lagrangian (14) reads

$$h(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\varphi(\mathbf{r}, t), \quad (18)$$

and indeed represents the total energy of the system, being the sum of the particle kinetic energy and the electric potential energy. Thus, from (4), the condition for the energy (18) to represent a constant of motion amounts to

$$0 = \frac{\partial L}{\partial t} = -q\frac{\partial \varphi}{\partial t} + \frac{q}{c}\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{A}}{\partial t}. \quad (19)$$

If the potentials do not depend explicitly on time, then the Lagrangian (14) is trivially invariant under time translation and (5) gives us a symmetry with $X \equiv 1$ and $\Psi_i \equiv 0$ whose conserved quantity (7) is precisely (18).

On the other hand, if the potentials carry an explicitly time dependence out, then (18) does not represent a constant of motion. However, still in this case, we may have a conserved quantity for (14) associated to time translational symmetry. Performing an infinitesimal time translation $t \rightarrow t' = t + \epsilon$ in (14), it changes by the amount

$$\delta L = \frac{\partial L}{\partial t}\delta t = \epsilon q \frac{\partial}{\partial t} \left(-\varphi + \frac{\dot{\mathbf{r}}}{c} \cdot \mathbf{A} \right). \quad (20)$$

Using the first equation of (15) it is possible to rewrite (20) as

$$\delta L = -\epsilon q [\dot{\varphi} + \mathbf{E} \cdot \dot{\mathbf{r}}]. \quad (21)$$

Thus if the electric power $q\mathbf{E} \cdot \dot{\mathbf{r}}$ transmitted by the field to the particle is null, we conclude from (7), with $G = -q\varphi$, that the particle kinetic energy $m\dot{\mathbf{r}}^2/2$ is a conserved quantity. This comprises, in other words, the well-known fact that the magnetic field cannot change the particle's speed.

Next we discuss spatial translational symmetry. Let $\mathbf{n}$ be a constant unit vector in three-dimensional space. Performing an infinitesimal spatial translation along the fixed $\mathbf{n}$ direction

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n}, \\ t' &= t. \end{aligned} \quad (22)$$

the Lagrangian (14) changes to

$$L' = L + q\nabla \left[-\varphi + \frac{\dot{\mathbf{r}}}{c} \cdot \mathbf{A} \right] \cdot \mathbf{n}\epsilon. \quad (23)$$

Once again we see we have two possibilities. First, if the potentials themselves enjoy translational invariance in the $\mathbf{n}$ direction, which means

$$\nabla \varphi \cdot \mathbf{n} = 0, \text{ and } \nabla A_i \cdot \mathbf{n} = 0, \quad (24)$$

then $L' = L$ and (7) gives us

$$C_{ST} = \left[m\dot{\mathbf{r}} + \frac{q}{c}\mathbf{A} \right] \cdot \mathbf{n} \quad (25)$$

as a conserved quantity. We interpret (25) as the $\mathbf{n}$ -component of the total linear momentum of the system, made up from the sum of the particle and field linear momenta along the $\mathbf{n}$ direction.

Notice that (25) is also the $\mathbf{n}$ projection of the Hamiltonian canonical momentum associated to (14).

The second possibility allows for L and L' to differ by a total time derivative. In this case we write $L' = L + \delta L$ in (23) and rearrange δL using (15) in order to obtain

$$\delta L = q \left(\mathbf{E} + \frac{\dot{\mathbf{r}}}{c} \times \mathbf{B} \right) \cdot \mathbf{n} \epsilon + \frac{d}{dt} \left(\frac{q}{c} \mathbf{A} \cdot \mathbf{n} \right) \epsilon. \quad (26)$$

Consequently, if the $\mathbf{n}$ -component of the Lorentz force applied to the particle is null, the transformed Lagrangian differs from the original one by a total time derivative $\dot{G}\epsilon$ with

$$G = \frac{q}{c} \mathbf{A} \cdot \mathbf{n}, \quad (27)$$

and (7) gives us the familiar result of particle linear momentum conservation

$$\frac{d}{dt} (m\dot{\mathbf{r}} \cdot \mathbf{n}) = 0. \quad (28)$$

We see that under a translational symmetry operation, if both the particle and the field remain invariant, then we have the system total linear momentum conservation as in (25); however, if the field does not remain invariant, then the particle linear momentum is conserved, as in (28), only if the applied electromagnetic force is null.

Spatial Rotations

Let us focus now on spatial rotations. Performing an infinitesimal rotation ϵ around a fixed axis $\mathbf{n}$, the particle coordinates transform as

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n} \times \mathbf{r}, \\ t' &= t, \end{aligned} \quad (29)$$

and the corresponding Lagrangian variation reads

$$\delta L = -q \left[\nabla \left(\phi - \frac{\dot{\mathbf{r}}}{c} \cdot \mathbf{A} \right) \right] \cdot (\mathbf{n} \times \mathbf{r}) \epsilon + \frac{q}{c} \mathbf{A} \cdot (\mathbf{n} \times \dot{\mathbf{r}}) \epsilon. \quad (30)$$

If it so happens that the system configuration is fully symmetric under arbitrary spatial rotations around $\mathbf{n}$, then δL in (30) is null and we have, as consequence from (7)-(8), the conservation of the total angular momentum projection

$$C_{SR} = \left[\mathbf{r} \times \left(m\dot{\mathbf{r}} + \frac{q}{c} \mathbf{A} \right) \right] \cdot \mathbf{n}. \quad (31)$$

If it is not the case that (30) vanishes identically, we may rewrite it as

$$\delta L = \epsilon \mathbf{r} \times \left(q\mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B} \right) \cdot \mathbf{n} + \epsilon \frac{d}{dt} \left[\frac{q}{c} (\mathbf{r} \times \mathbf{A}) \cdot \mathbf{n} \right]. \quad (32)$$

Once more we have used the electromagnetic field equations (15) to obtain (32) from (30). From (32) we infer that, if the torque due to the external electromagnetic field on the particle is null, the variation of the Lagrangian is a total time derivative. In this case, with $G = \frac{q}{c} (\mathbf{r} \times \mathbf{A}) \cdot \mathbf{n}$, Noether's theorem gives us

$$\frac{d}{dt} [(\mathbf{r} \times m\dot{\mathbf{r}}) \cdot \mathbf{n}] = 0. \quad (33)$$

So we see that null external torque leads to the particle angular momentum conservation even if the field configuration does not respect rotational symmetry.

Galilean Boosts

The previous transformations all involved only rest reference frames. Now let us consider a change of coordinates to a inertial frame moving with constant velocity relative to the first, along a fixed $\mathbf{n}$ direction. Since we are not yet considering the relativistic case we describe it by a Galilean infinitesimal transformation

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n} t, \\ t' &= t.\end{aligned}\quad (34)$$

The infinitesimal parameter ϵ represents the relative velocity between the two inertial frames. The system Lagrangian (14) transforms under (34) as

$$\delta L = \epsilon \frac{d}{dt} \left[\left(\frac{q}{c} \mathbf{A} t + m \mathbf{r} \right) \cdot \mathbf{n} \right] + \epsilon \left(q \mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B} \right) \cdot \mathbf{n} t. \quad (35)$$

If the $\mathbf{n}$ -component of the external force is zero, the Lagrangian variation (35) is a total time derivative. Noether's theorem then leads, from (7), to the center of mass position conservation

$$C_{GB} = (-m \dot{\mathbf{r}} t + m \mathbf{r}) \cdot \mathbf{n}, \quad (36)$$

which in turn gives an immediate solution for the particle vector position projection along $\mathbf{n}$

$$(\mathbf{r} \cdot \mathbf{n}) = (\dot{\mathbf{r}} \cdot \mathbf{n}) t + (\mathbf{r}_0 \cdot \mathbf{n}) \quad (37)$$

for some constant $\mathbf{r}_0$. This simply signals consistently the lack of acceleration for the particle along the $\mathbf{n}$ direction.

4 Relativistic Particle in External Electromagnetic Field

In this section we generalize the previous analyses to the relativistic case. Obviously the kinetic energy term in Lagrangian (14) is not invariant under a general Lorentz transformation and so is not consistent with the special theory of relativity. We can write a Lagrangian for the relativistic particle embedded in an external electromagnetic field as

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = -mc^2 \sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} - q\varphi + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}, \quad (38)$$

where once more $\varphi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials.

The Euler-Lagrange equations corresponding to (38) read

$$\frac{d}{dt} \left(\frac{m \dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} \right) + \frac{q}{c} \dot{\mathbf{A}} = -q \nabla \left(\varphi - \frac{\dot{\mathbf{r}} \cdot \mathbf{A}}{c} \right), \quad (39)$$

or simply

$$\frac{d}{dt} \left(\frac{m \dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} \right) = q \mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B}, \quad (40)$$

by means of (15). Equation (40) is the relativistic generalization of (17) and can be interpreted as the relativistic Newton's second law of motion justifying thus the use of (38).

In fact (38) is written for a particular reference frame and is not Lorentz invariant. However, including integration along time, the corresponding action can be written as

$$S = \int dt L = \int d\tau \left\{ -mc^2 - \frac{q}{c} U^\mu \cdot A_\mu \right\} \quad (41)$$

being τ the proper time and $U^\mu = (\gamma c, \gamma \dot{\mathbf{r}})$ the relativistic quadrivelocity. Action (41) is manifestly Lorentz invariant.

We are now ready to look for other conserved quantities associated to (38) through Noether's Theorem.

Time and Spatial Translations

A direct calculation of the energy function (3) for (38) leads to the expression

$$h(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{mc^2}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} + q\varphi. \quad (42)$$

As we have already pointed out, if the Lagrangian does not depend explicitly on time, the energy function produces a constant of motion. In the present situation this happens if the potentials ϕ and $\mathbf{A}$ in (38) do not exhibit an explicit time dependence; in this case the total energy of the system (42) is conserved.

If the potentials do have an explicitly time dependence then an infinitesimal translation in time $t \rightarrow t' + \epsilon$ leads again to equations (20) and (21). Following the same line of reasoning, we conclude that if the transmitted electric power $\mathbf{E} \cdot \dot{\mathbf{r}}$ is null then we have, from (7), the conservation of the quantity

$$E = \frac{mc^2}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} \quad (43)$$

which represents the relativistic particle's energy.

Concerning spatial translations, if (38) remains invariant under an infinitesimal displacement along a fixed $\mathbf{n}$ direction

$$\mathbf{r}' = \mathbf{r} + \epsilon \mathbf{n}, \quad (44)$$

the corresponding Noether conserved quantity (7) reads

$$C_{RST} = \left[\frac{m\dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} + \frac{q}{c} \mathbf{A} \right] \cdot \mathbf{n}. \quad (45)$$

That means that the total relativistic linear momentum of the particle-plus-field system is conserved along the $\mathbf{n}$ direction. Similarly to the non-relativistic case, even if (38) is not fully invariant under spatial translations, equations (23) and (26) hold. Assuming the $\mathbf{n}$ -component of the Lorentz force to be null we are led through (7), with G given by (27), to the conservation of the particle relativistic momentum along the $\mathbf{n}$ -direction

$$\frac{d}{dt} \left[\frac{m\dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} \cdot \mathbf{n} \right] = 0. \quad (46)$$

Equations (45) and (46) are respectively the relativistic generalizations of (25) and (28).

Spatial Rotations

Let us perform a spatial rotation (29) in the relativistic system (38). If the Lagrangian (38) enjoys full rotational invariance it immediately follows from (7) that the generalization of (31) reads

$$C_{RSR} = \left[\mathbf{r} \times \left(\frac{m\dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} + \frac{q}{c} \mathbf{A} \right) \right] \cdot \mathbf{n}, \quad (47)$$

which means the $\mathbf{n}$ -projection of the total relativistic momentum of the particle and field is conserved. Furthermore, if (38) is not rotational invariant, a simple calculation shows that (30) and (32) read exactly the same and express the variation of (38) under (29). Then, if the electromagnetic torque applied by the field on the particle is null, the Lagrangian variation (32) is a total time derivative and (33) generalizes to

$$\frac{d}{dt} \left[\left(\mathbf{r} \times \frac{m\dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} \right) \cdot \mathbf{n} \right] = 0, \quad (48)$$

expressing now the conservation of only the particle angular momentum along the $\mathbf{n}$ -direction.

Lorentz Boosts

A Lorentz boost is a homogeneous Lorentz transformation connecting two inertial frames with constant relative velocity and no spatial rotation. In usual relativistic notation an homogeneous infinitesimal Lorentz transformation can be written as

$$x^\alpha \rightarrow x'^\alpha = (\delta^\alpha_\beta + \omega^\alpha_\beta) x^\beta, \quad (49)$$

being ω^α_β antisymmetric and representing six real independent infinitesimal parameters corresponding to three degrees of freedom describing ordinary spatial rotations and three describing Lorentz boosts. In the standard vectorial notation we have been using, an infinitesimal Lorentz boost along the spatial $\mathbf{n}$ direction can be written as

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n} t, \\ t' &= t + \frac{\epsilon}{c^2} (\mathbf{n} \cdot \mathbf{r}), \end{aligned} \quad (50)$$

where the infinitesimal parameter ϵ represents the relative velocity between the two frames. Clearly (50) is the relativistic generalization of the infinitesimal Galilean boost (34). Since the transformation (50) explicitly changes the time, it is not enough to check the Lagrangian variation - rather we should study the action variation (6) under (50) which reads

$$\Delta S = \epsilon \int_{t_1}^{t_2} \left[\left(q\mathbf{E} + q\frac{\dot{\mathbf{r}}}{c} \times \mathbf{B} \right) \cdot \mathbf{n} t + q(\dot{\mathbf{r}} \cdot \mathbf{E}) \left(-\frac{\mathbf{n} \cdot \mathbf{r}}{c^2} \right) + \frac{d}{dt} \left[\frac{q}{c} \mathbf{A} \cdot \mathbf{n} t - q\phi \left(\frac{\mathbf{n} \cdot \mathbf{r}}{c^2} \right) \right] \right] \quad (51)$$

In case of null external $\mathbf{n}$ -projection force and null transmitted electric power the variation ΔS is a total time derivative and (7) gives us the conserved quantity

$$C_{LB} = m \left(1 - \frac{\dot{\mathbf{r}}^2}{c^2} \right)^{-1/2} (\mathbf{r} - \dot{\mathbf{r}} t) \cdot \mathbf{n} \quad (52)$$

which generalizes (36).

5 Conclusion

Noether's theorem connects continuous symmetries of the classical action to conserved quantities. We have reviewed and proved Noether's theorem at classical level. We have applied Noether's theorem to a system made up of a particle interacting with an external electromagnetic field, considering both relativistic and nonrelativistic cases. We have discussed the consequences of all space-time symmetries of the action, calculating the corresponding conserved quantities. We have seen that when a space-time transformation is performed in the particle's space-time coordinates, the field's behavior under the transformation may lead to different results, namely to conservation of quantities either related solely to the particle or to the whole configuration set, particle plus field.

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