
Supersymmetric Aspects of One Dimensional Quantum Mechanics

Aspectos Supersimétricos de Mecânica Quântica Unidimensional

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Abstract

We review SUSY in non-relativistic quantum mechanics as a powerful tool for studying one dimensional potentials and discuss some applications. The Dirac operatorial factorization method for the harmonic oscillator in QM is generalized to a broader scope. A Schrödinger second order operator can be factorized into two first-order ones by solving a Riccati equation and determining the corresponding superpotential. We apply the technique to square well, Pöschl-Teller and finite barrier potentials, analyzing the resulting energy spectra and the transmission and reflection coefficients. Both discrete and continuous energy spectrum cases are discussed.

Keywords Supersymmetry, supersymmetric quantum mechanics, Pöschl-Teller potential, SUSY, factorization method.

Resumo

Revisamos SUSY em mecânica quântica não-relativista como uma ferramenta poderosa para o estudo de potenciais unidimensionais e discutimos algumas aplicações. O método operatorial da fatoração de Dirac para o oscilador harmônico na mecânica quântica é generalizado para um maior escopo e abrangência. Um operador de Schrödinger de segunda ordem pode ser fatorado em dois de primeira ordem resolvendo uma equação de Riccati e determinando o correspondente superpotencial. Aplicamos a técnica para os potenciais de poço quadrado, Pöschl-Teller e de barreira finita, analisando os espectros de energia resultantes e os coeficientes de transmissão e reflexão. Ambos os casos de espectro de energia, discreto e contínuo, são abordados.

Palavras-chave Supersimetria, mecânica quântica supersimétrica, potencial de Pöschl-Teller, SUSY, método da fatoração.

1 Introduction

In recent years, the role of supersymmetry (SUSY) in non-relativistic quantum mechanics (QM) has been extensively analysed, leading to a consistent classification of interacting potentials [1, 2, 3, 4]. First considered as an essential ingredient for any fundamental interactions unifying theory, SUSY has firmly established itself as an important mathematical technique for approaching problems, both in quantum mechanics and in quantum field theory, in its own right. Particularly, SUSY has been successfully used as a powerful method for analytically solving the Schrödinger equation for some potentials as well as for constructing approximation

methods for handling more involved not exactly solvable ones, establishing thus the eigenvalues and eigenvectors properties of the corresponding Schrödinger operators.

The main idea in SUSY quantum mechanics is to relate the Hamiltonian H^- for a certain problem to another one H^+ , known as the former's SUSY partner. This relation is achieved by means of introducing a superpotential $W(x)$ in the theory. SUSY connects the eigenfunctions of the two partner Hamiltonians in a simple way – the knowledge of one set of eigenfunctions permits one to directly calculate the other. Particularly, the SUSY partner potentials are isospectral. This leads to a symmetry (the supersymmetry) relating eigenfunctions of the distinct Hamiltonians H_- and H_+ with the same energy.

In section 2 below we review the basic ingredients of SUSY, fixing our notations and conventions. We define the SUSY partner Hamiltonian $H_{\pm}$ and the superpotential $W(x)$ as well as the key operators $A^{\pm}$ establishing some important relations and properties. In sections 3 and 4 we discuss respectively the infinite and finite square well potentials, calculating the corresponding SUSY partner potentials with its eigenenergies and eigenfunctions. Particularly we see that the Pöschl-Teller potential [5] is the SUSY partner of the square well potential. In section 5 we discuss the case of continuous spectra, relating the partner potentials reflection and transmission coefficients and exemplifying the results with some barrier potentials.

2 Supersymmetry in Quantum Mechanics

Consider a spinless mass m particle on a line subjected to a one dimensional real potential $V(x)$. Its quantum mechanical description amounts to constructing an infinite dimensional Hilbert space $\mathcal{E}$ of kets $|\psi\rangle$ representing the possible particle states. The particle dynamical evolution is governed by the Hamiltonian

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}), \quad (1)$$

where $\mathbf{P}$ and $\mathbf{X}$ are, correspondingly, the momentum and position Hermitian operators acting on $\mathcal{E}$. Since the potential is time independent, the well-known separation of variables technique can be applied, leading, in the position basis, to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x). \quad (2)$$

This is an ordinary second order differential equation for the complex wave function $\psi(x)$ corresponding to the ket $|\psi\rangle \in \mathcal{E}$. We may also interpret (2) as an eigenvalue-eigenvector problem. Given a potential $V(x)$, we seek for complex eigenfunctions $\psi(x)$ and corresponding real eigenvalues E . The real numbers E , being eigenvalues of the Hamiltonian, represent the energy spectrum of the theory.

Let $\psi_0(x)$ be the ground state solution of (2), corresponding to the minimal energy E_0 . Redefining the potential as $V_-(x) \equiv V(x) - E_0$ we may write

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_0(x) + V_-(x)\psi_0(x) = 0 \quad (3)$$

for the ground state and all energy levels get downshifted by E_0 . Associated to the potential V_- we can define the corresponding Hamiltonian H^- given by

$$H^- \equiv -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-. \quad (4)$$

Labeling eigenfunctions and eigenvalues by the subscript n we have explicitly

$$H^- \psi_n^-(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n^-(x) + V_-(x)\psi_n^-(x) = E_n^- \psi_n^-(x), \quad (5)$$

with $E_n^- \equiv E_n - E_0$. As defined above the ground state of H^- can be readily checked to have zero energy

$$H^- \psi_0^- = 0. \quad (6)$$

Naturally the eigenstates of (2) are the same as those of (5) and particularly $\psi_0 = \psi_0^-$.

Aiming to obtain a supersymmetric partner for the potential V_- , we shall now introduce the elements of supersymmetry in the theory. Inspired by the well-known creation/annihilation operator technique of the harmonic oscillator we begin factorizing the second order operator H^- into

$$H^- = A^+ A^-, \quad (7)$$

with,

$$\begin{aligned} A^- &\equiv \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \\ A^+ &\equiv \frac{-\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \end{aligned} \quad (8)$$

where $W(x)$ is a solution of the Riccati non-linear first order differential equation

$$V_- = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \quad (9)$$

The quantity $W(x)$ is called the superpotential associated to the original potential $V(x)$ in (2) and satisfies the commutation relation

$$[A^-, A^+] = \frac{2\hbar}{\sqrt{2m}} W'(x). \quad (10)$$

Notice that if a ground state eigenfunction ψ_0 satisfying (3) for a particular one-dimensional potential $V(x)$ is known, one can immediately write

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)} \quad (11)$$

as a solution to the (9). Switching the order between A^- and A^+ in (8) we define the operator

$$H^+ \equiv A^- A^+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+, \quad (12)$$

with

$$V_+ \equiv \frac{\hbar}{\sqrt{2m}} W' + W^2. \quad (13)$$

Here H^+ and V_+ are known respectively as the SUSY partners of H^- and V_- . As can be easily checked, A^+ and A^- are the adjoint of each other, while both Hamiltonians H^+ and H^- are Hermitian semi-positive-definite operators. In the following, let us figure out how the eigenvalues and eigenfunctions of H^- and H^+ are interrelated. Denoting the eigenfunctions of H^- (H^+) by ψ_n^- (ψ_n^+) we write

$$\begin{aligned} H^- \psi_n^-(x) &= E_n^- \psi_n^-(x), \\ H^+ \psi_n^+(x) &= E_n^+ \psi_n^+(x). \end{aligned} \quad (14)$$

Concerning solutions $\phi^\pm$ to $A^+\phi^+ = A^-\phi^- = 0$, we may write

$$\phi^\pm \sim \exp \left[\mp \frac{\sqrt{2m}}{\hbar} \int W(x) dx \right], \quad (15)$$

and particularly $\phi^+ \sim (\phi^-)^{-1}$. That means if ϕ^- is normalizable, ϕ^+ is not. We assume ϕ^- to be normalizable and consider $\phi^- = \psi_0^-$ which satisfies

$$H^-\psi_0^- = A^+(A^-\psi_0^-) = 0. \quad (16)$$

Explicitly we write

$$\psi_0^-(x) = C \exp \left[-\frac{\sqrt{2m}}{\hbar} \int W(x) dx \right], \quad (17)$$

with

$$\int dx |\psi_0^-|^2 = 1. \quad (18)$$

Therefore, considering the eigenvalues in (14) ordered by increasing value of energies, we must have $E_0^- = 0$ and $E_0^+ > 0$. Observing that

$$H^+(A^-\psi_n^-) = E_n^-(A^-\psi_n^-), \quad (19)$$

and comparing with the second equation of (14) we see that (i) the spectrum of H^+ coincides with that of H^- with the sole exception of $E_0^- = 0$ and (ii) the eigenfunctions of H^+ are proportional to $A^-\psi_n^-$. We thus write

$$E_n^+ = E_{n+1}^-, \quad n \geq 0, \quad (20)$$

and

$$\psi_n^+ = \frac{1}{\sqrt{E_{n+1}^-}} A^-\psi_{n+1}^-, \quad n \geq 0. \quad (21)$$

By applying A^+ to both sides of the last equation it can be inverted to

$$\psi_{n+1}^- = \frac{1}{\sqrt{E_n^+}} A^+\psi_n^+, \quad n \geq 0. \quad (22)$$

We see that the A^- and A^+ operators connect H^- and H^+ eigenstates with the same energy. Knowledge of the eigenstates and eigenvalues of one of the Hamiltonians $H^\pm$ leads to the knowledge of the corresponding solution for its partner. In the following sections we apply this formalism to specific one dimensional potentials.

3 Infinite Square Well Potential

In this section we illustrate the previously discussed central SUSYQM ideas in the simple infinite square well potential, also known as “particle in a box potential”. We start with the time-independent Schrödinger equation (2) with the potential $V(x)$ given by

$$V(x) = \begin{cases} 0, & |x| \leq a; \\ \infty, & |x| > a. \end{cases} \quad (23)$$

The positive real parameter a , with length dimension, characterizes the well potential width. The wave function vanishes for $|x| \geq a$ confining thus the particle inside the “box” $|x| < a$. For $|x| < a$, equation (2) reduces to

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x). \quad (24)$$

Non-positive energy eigenvalues lead to wave solutions which cannot match continuity at $|x| = a$, unless $\psi \equiv 0$ which is not an allowed eigenvector by definition. Therefore we must have $E > 0$. Defining $k = \sqrt{\frac{2mE}{\hbar^2}}$ we write the general solution for (24) as

$$\psi(x) = A \cos kx + B \sin kx. \quad (25)$$

The boundary condition $\psi(a) = \psi(-a) = 0$ enforces either $B = 0$ with $ka = (2n + 1)\pi/2$ or $A = 0$ with $ka = n\pi$ for natural n^1 .

Therefore, labelling the solutions by $n \in \mathbb{N}$ in increasing order of energy value, we have

$$\begin{aligned} E_n &= \frac{\pi^2 \hbar^2}{8ma^2} n^2, \quad n = 1, 2, 3, \dots, \\ \psi_n &= \begin{cases} B_n \cos \left[\frac{n\pi x}{2a} \right], & \text{for } n = 1, 3, 5, \dots, \\ B_n \sin \left[\frac{n\pi x}{2a} \right], & \text{for } n = 2, 4, 6, \dots, \end{cases} \end{aligned} \quad (26)$$

By subtracting the ground state energy and shifting n to $n + 1$ we get

$$\begin{aligned} E_n^- &= \frac{\pi^2 \hbar^2}{8ma^2} n(n + 2), \quad n = 0, 1, 2, \dots, \\ \psi_n^- &= \begin{cases} C_n \cos \left[\frac{(n+1)\pi x}{2a} \right], & \text{for } n = 0, 2, 4, \dots, \\ C_n \sin \left[\frac{(n+1)\pi x}{2a} \right], & \text{for } n = 1, 3, 5, \dots \end{cases} \end{aligned} \quad (27)$$

according to the previous SUSY notation. The superpotential can be readily obtained from (11) as

$$W(x) = \frac{\hbar\pi}{\sqrt{8ma^2}} \tan \left(\frac{\pi x}{2a} \right), \quad (28)$$

and the SUSY partner potential (13) reads

$$V_+(x) = \frac{\hbar^2\pi}{8ma^2} \left[2 \sec^2 \left(\frac{\pi x}{2a} \right) - 1 \right]. \quad (29)$$

The potential $V_+(x)$ in (29) can be promptly recognized as the Pöschl-Teller potential first introduced in [5].

Now we may use our knowledge of the solution to the infinite square well potential (27) and its corresponding superpotential (28) to generate the set of solutions (22) to the Pöschl-Teller potential (29). For instance, for the first three eigenfunctions of H^+ , an explicit calculation using (22) leads to

$$\begin{aligned} \psi_0^+ &\sim \cos^2 \left(\frac{\pi x}{2a} \right) \\ \psi_1^+ &\sim \sin \left(\frac{\pi x}{a} \right) \cos \left(\frac{\pi x}{2a} \right) \\ \psi_2^+ &\sim 4 \cos^4 \left(\frac{\pi x}{2a} \right) - 5 \sin^2 \left(\frac{\pi x}{a} \right) \end{aligned} \quad (30)$$

¹Once again we exclude the trivial nonrenormalizable case $\psi(x) \equiv 0$.

Further, the corresponding eigenvalues, obtained from (20), are easily found to be $E_0^+ = \frac{3\pi^2\hbar^2}{8ma^2}$, $E_1^+ = \frac{8}{3}E_0^+$ and $E_2^+ = 5E_0^+$.

Thus we see in this example that knowledge of the solution of the simpler eigenvalue problem for H^- enables one to readily solve the more involving Pöschl-Teller potential problem (29).

4 Finite Square Well Potential

Let us now consider the so-called finite square well potential defined as

$$V(x) = \begin{cases} 0, & |x| \leq a; \\ V_0, & |x| > a. \end{cases} \quad (31)$$

where the positive constant V_0 is the depth of the well. This can be considered as a generalization of the case treated in the preceding section whereas now the well has finite potential walls. Since we are interested in bound states, in this section we shall restrict ourselves to the case $E < V_0$. Unbounded states will be studied on next section. Classically, if $E < V_0$, we expect the particle to be found only within the region $|x| \leq a$. However, quantum mechanically, unlike in the infinite well case, there is a non-zero probability for finding the particle outside the box, even when $E < V_0$.

For the region inside the box, $V(x) = 0$, the time independent Schrödinger equation (2) reduces once more to (24) which is easily solvable. However, both the wave function and its first derivative must be continuous in the whole real line. We claim we have acceptable solutions only for positive E . Assuming $E < 0$ we define the reals

$$\rho_1 = \sqrt{2m(V_0 - E)}/\hbar, \quad \rho_2 = \sqrt{-2mE}/\hbar, \quad (32)$$

and write a tentative overall solution as

$$\psi(x) = \begin{cases} Ae^{\rho_1 x}, & x \leq -a, \\ Be^{\rho_2 x} + Ce^{-\rho_2 x}, & |x| \leq a, \\ De^{-\rho_1 x}, & x \geq a. \end{cases} \quad (33)$$

Continuity of (33) and its first derivative at $x = \pm a$ leads to the consistency condition

$$\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^2 = e^{4\rho_2 a} \quad (34)$$

which cannot be satisfied by any pair of positive real numbers ρ_1 and ρ_2 . Similarly, assuming $E = 0$, we write

$$\psi(x) = \begin{cases} Ae^{\rho_1 x}, & x \leq -a, \\ Bx + C, & |x| \leq a, \\ De^{-\rho_1 x}, & x \geq a, \end{cases} \quad (35)$$

and obtain the consistency condition

$$1 + a\rho_1 = 0 \quad (36)$$

which leads once more to a contradiction. Therefore there are no energy eigenvalues satisfying $E \leq 0$ associated to the potential (31).

Concerning $0 < E < V_0$ we shall prove in the following, by explicit construction, that there do exist wave function solutions satisfying the continuity requirements. Indeed, for $0 < E < V_0$ we write

$$\psi(x) = \begin{cases} Ae^{k'x}, & x \leq -a, \\ Be^{ikx} + Ce^{-ikx}, & |x| \leq a, \\ De^{-k'x}, & x \geq a. \end{cases} \quad (37)$$

with $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $k' = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$. Imposing continuity of $\psi(x)$ and $\psi'(x)$ at $x = \pm a$ we obtain the relations

$$B = \frac{k' + ik}{2ik} e^{-(k'+ik)a} A, \quad C = \frac{ik - k'}{2ik} e^{-(k'+ik)a} A, \quad D = \frac{k'^2 + k^2}{2kk'} \sin(2ka) A, \quad (38)$$

and the consistency condition

$$\left(\frac{k' - ik}{k' + ik} \right)^2 = e^{4ika}. \quad (39)$$

Contrary to (34) and (36), in the present case it is possible to find energy eigenvalues satisfying (39). In fact, (39) leads to the two possibilities

$$k' = k \tan(ka) \quad \text{or} \quad k = -k' \tan(ka), \quad (40)$$

restricting the energy eigenvalues to a countable set. There exists always at least one solution to (40) and consequently to (39). To see this, we define $\Theta \equiv a\sqrt{k^2 + k'^2} = a\sqrt{2mV_0}/\hbar$, $\theta \equiv ka$ and the real functions

$$p(\theta) = \theta \tan(\theta), \quad \bar{p}(\theta) = -\theta \cot(\theta), \quad q(\theta) = \sqrt{\Theta^2 - \theta^2}. \quad (41)$$

The first equation of (40) is equivalent to $p(\theta) = q(\theta)$. Observe that $p(\theta)$ is monotonic increasing in $[0, \frac{\pi}{2})$ with $\lim_{\theta \rightarrow \frac{\pi}{2}} p(\theta) = \infty$ and therefore defines a bijection between $[0, \frac{\pi}{2})$ and $\mathbb{R}^+$. On the other hand, for a given $\Theta > 0$, the restriction of $q(\theta)$ to $[0, \Theta]$ is a bounded from above monotonic decreasing bijection. Thus for all $\Theta > 0$ there exists a unique $\theta_0 \in [0, \min(\frac{\pi}{2}, \Theta))$ such that $p(\theta_0) = q(\theta_0)$. This determines a minimal energy eigenvalue

$$E_0 = \frac{\hbar^2 \theta_0^2}{2ma^2} \quad (42)$$

in the interval $0 < E_0 < V_0$. The second equation in (40) is equivalent to $\bar{p}(\theta) = q(\theta)$ and exhibits solutions only if $\Theta \geq \frac{\pi}{2}$.

The precise number of energy eigenvalues E satisfying (40) will depend on the values of the parameters a , V_0 and m , through the combination Θ , and is given by the least natural N such that

$$\Theta \leq N\pi. \quad (43)$$

For each solution of (40) k_i, k'_i , with $i = 0, \dots, N-1$, we have an energy eigenvalue

$$E_i = \frac{\hbar^2 k_i^2}{2m}, \quad (44)$$

and a corresponding eigenfunction (37) which upon substitution of (38) into (37) may be rewritten as

$$\psi_i(x) = \begin{cases} Ae^{k'_i x}, & x \leq -a, \\ e^{-k'_i a} \left[\frac{k'_i}{k_i} \sin[k_i(a+x)] + \cos[k_i(a+x)] \right], & |x| \leq a, \\ \frac{k_i'^2 + k_i^2}{2k_i k'_i} \sin(2k_i a) Ae^{-k'_i x}, & x \geq a. \end{cases} \quad (45)$$

As usual, we label the eigenvalues (44) in increasing order of energy. Thus, for even i , k_i, k'_i is a solution of the first equation in (40), whilst for odd i , the pair k_i, k'_i is a solution of the second. Using (40), equation (45) can be further simplified to

$$\psi_n^-(x) = \begin{cases} Ae^{k'_n x}, & x \leq -a, \\ \frac{e^{-k'_n a} A}{\cos k_n a} \cos k_n x, & |x| \leq a, \\ Ae^{-k'_n x}, & x \geq a. \end{cases} \quad (46)$$

for n even and

$$\psi_n^-(x) = \begin{cases} Ae^{k'_n x}, & x \leq -a, \\ -\frac{e^{-k'_n a} A}{\sin k_n a} \sin k_n x, & |x| \leq a, \\ -Ae^{-k'_n x}, & x \geq a. \end{cases} \quad (47)$$

for n odd. The eigenfunctions $\psi_n^-(x)$ have a definite parity which coincides with that of the natural index n . One can easily check that both (46) and (47), as well as its first derivatives, fulfill the continuity requirement in the whole real line.

In order to apply the technique developed in section 2 we redefine the potential and energy levels by subtracting the minimal energy eigenvalue (42) as

$$V_-(x) = \begin{cases} -\frac{\hbar^2 k_0^2}{2m}, & |x| \leq a; \\ V_0 - \frac{\hbar^2 k_0^2}{2m}, & |x| > a, \end{cases} \quad (48)$$

$$E_n^- = \frac{\hbar^2}{2m}(k_n^2 - k_0^2). \quad (49)$$

The corresponding eigenfunctions are precisely given by (46) and (47). The superpotential, given by (11), is found to read

$$W(x) = \begin{cases} -\frac{\hbar k'_0}{\sqrt{2m}}, & x \leq -a; \\ \frac{\hbar k_0}{\sqrt{2m}} \tan(k_0 x), & |x| > a; \\ \frac{\hbar k'_0}{\sqrt{2m}}, & x \geq a \end{cases} \quad (50)$$

and the corresponding SUSY partner potential reads

$$V_+ = \begin{cases} \frac{\hbar^2 k_0^2}{2m} [2 \sec^2(k_0 x) - 1], & |x| \leq a; \\ \frac{\hbar^2 k_0'^2}{2m}, & |x| > a. \end{cases} \quad (51)$$

5 Continuous Spectra

In this section we shall relate the reflection and transmission coefficients of SUSY partner potentials with continuous spectra.

Up to now we have been tacitly assuming discrete spectra for both Hamiltonians H_- and H_+ . However, if we simply consider n to be a continuous label, rather than a discrete one, the same previous analysis goes through, step by step, until Equation (19). Since we do not have a “next level” in the continuous case, equations (20) and (22) are meaningless and useless. Despite of that, (19) tells us that operators $A^\pm$ connect $H_\pm$ eigenfunctions with the same eigenvalue. Changing notation slightly, from

$$\begin{aligned} H^- \psi_{(E^-)}^- &= E^- \psi_{(E^-)}^-, \\ H^+ \psi_{(E^+)}^+ &= E^+ \psi_{(E^+)}^+, \end{aligned} \quad (52)$$

it follows that

$$\begin{aligned} H^+ (A^- \psi_{(E^-)}^-) &= E^- (A^- \psi_{(E^-)}^-), \\ H^- (A^+ \psi_{(E^+)}^+) &= E^+ (A^+ \psi_{(E^+)}^+), \end{aligned} \quad (53)$$

which particularly means that wave functions associated to SUSY partner potentials with the same energy $E^- = E^+$ are connected through $\psi^- \sim A^+ \psi^+$ and $\psi^+ \sim A^- \psi^-$. Since we wish to discuss scattering, we assume that both potentials remain finite in the limit $x \rightarrow \pm\infty$ and define

$$\begin{aligned}\lim_{x \rightarrow +\infty} W(x) &= W_L, \\ \lim_{x \rightarrow -\infty} W(x) &= W_R.\end{aligned}\tag{54}$$

From (9) and (13) we get

$$\begin{aligned}\lim_{x \rightarrow -\infty} V_{\pm}(x) &= W_L^2, \\ \lim_{x \rightarrow +\infty} V_{\pm}(x) &= W_R^2.\end{aligned}\tag{55}$$

Consider an incident plane wave e^{ikx} coming from $x = -\infty$. Scattering off the potentials $V_{\pm}(x)$ results in transmitted and reflected waves such that the asymptotic wave functions read

$$\begin{aligned}\psi_{\pm}(-\infty) &\sim e^{ikx} + R_{\pm}e^{-ikx}, \\ \psi_{\pm}(+\infty) &\sim T_{\pm}e^{ik'x},\end{aligned}\tag{56}$$

where $R_{\pm}$ and $T_{\pm}$ are respectively the reflection and transmission coefficients and k and k' are wave numbers given by $k = [2m(E - W_L^2)/\hbar^2]^{1/2}$ and $k' = [2m(E - W_R^2)/\hbar^2]^{1/2}$. Using now $\psi^- \sim A^+ \psi^+$ with A^+ given by (8) we relate the reflection and transmission coefficients of the two potentials as

$$\begin{aligned}R_- &= \frac{W_L + ik\hbar/\sqrt{2m}}{W_L - ik\hbar/\sqrt{2m}}R_+, \\ T_- &= \frac{W_R - ik'\hbar/\sqrt{2m}}{W_L - ik\hbar/\sqrt{2m}}T_+.\end{aligned}\tag{57}$$

Concerning the probability amplitudes, observe that $|R_-|^2 = |R_+|^2$ as well as $|T_-|^2 = |T_+|^2$. Thus, partner potentials have the same reflection and transmission probabilities.

Now we continue our study analyzing the motion of a particle in a potential $V(x)$ which has the form of a rectangular barrier given by

$$V(x) = \begin{cases} V_0, & 0 < x < a; \\ 0, & \text{otherwise.} \end{cases}\tag{58}$$

The potential barrier is commonly used in Applied Quantum Mechanics for studying “quantum tunneling” effects. Here we will consider the case where the particle energy is less than V_0 . Solving the time independent Schrödinger equation for this potential we obtain

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < 0, \\ Ce^{k'x} + De^{-k'x}, & 0 < x < a, \\ Fe^{ikx}, & x > a, \end{cases}\tag{59}$$

where, $k^2 = \frac{2mE}{\hbar^2}$ and $k'^2 = \frac{2m(V_0-E)}{\hbar^2}$. The five constants can be obtained requiring that both $\psi(x)$ and $\psi'(x)$ be continuous at $x = 0$ and $x = a$. These constants will give us the reflection and transmission coefficients, R and T, respectively. By using the wave solution inside the barrier we can obtain the superpotential as

$$W(x) = -\frac{\hbar k'}{\sqrt{2m}} \tanh(k'x).\tag{60}$$

Take a closer look at equations (28) and (50) for bound states, we can note that there is a certain similarity among those and the superpotential for a continuous spectra, as showed in (60). For the SUSY partner we have

$$V_+ = \frac{\hbar^2 k'^2}{2m} [2\text{sech}^2(k'x) - 1] . \quad (61)$$

By the same token as the superpotential, the SUSY partner has a certain similarity as those ones showed in equations (29) and (51).

6 Conclusion

We have reviewed SUSY in non-relativistic quantum mechanics. We have discussed the Schrödinger equation in the supersymmetry context for one-dimensional potentials, introducing the concepts of superpotential for both discrete and continuous cases. We have worked out the examples of square well, Pöschl-Teller and barrier potentials. SUSY proved to be a powerful analysis method for such one dimensional problems, connecting the solutions of the Schrödinger equation corresponding to two partner potentials. Shape invariance and other one dimensional potentials are currently under investigation with the SUSY analysis tools introduced here.

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