

Value Functions on Quaternion Algebras Funções de Valor em Álgebras de Quatérnios

Maurício A. Ferreira

Departamento de Ciências Exatas, UEFS, Feira de Santana, BA, Brasil
maferreira@uefs.br

Abstract

A value function is a map like a valuation that can be defined on rings with zero divisors. In this paper, we deal with a new type of value function for finite-dimensional semisimple algebras over valued fields called gauge. While a valuation on a field does not always extend to a valuation on a quaternion division algebra, we show how to construct explicitly a gauge in an arbitrary quaternion algebra. We also prove that the associated graded algebra of the gauge constructed here is a graded quaternion algebra.

Keywords Valuation theory, division rings, simple algebras, quaternion algebra, graded algebras.

Resumo

Uma função de valor é uma aplicação, como uma valorização, que pode ser definida em anéis com divisores de zero. Neste artigo, nós lidamos com um novo tipo de função de valor para álgebras semi-simples de dimensão finita sobre corpos valorizados, chamada gauge. Enquanto uma valorização em um corpo nem sempre se estende para uma valorização em uma álgebra de quatérnios de divisão, nós mostramos como construir explicitamente uma gauge em uma álgebra de quatérnios arbitrária. Além disso, nós provamos que a álgebra graduada associada a gauge construída aqui é uma álgebra de quatérnios graduada.

Palavras-chave Teoria de valorização, anéis de divisão, álgebras simples, álgebras de quatérnios, álgebras graduadas.

1 Introduction

Valuation theory on noncommutative division rings had a substantial development in the last three decades. This theory has proved very useful in constructing counterexamples for open problems in noncommutative ring theory, as noncrossed product division algebras and division algebras with nontrivial SK_1 . See [10] for a complete review on this subject. On the other hand, this rich theory has its limitations. Valuations on division rings do not behave well with respect to extensions. Given a valuation v on a subring E of a division ring D , there is often no valuation on D extending v (see Example 3.3). Also, valuations can not be defined on rings with zero divisors. Even when one starts out with an F -division algebra D with a valuation, very often one needs to consider a scalar extension $D \otimes_F L$ for a field extension L of F . In general, $D \otimes_F L$ is a simple algebra, but need not be a division algebra. This has led to efforts to generalize valuations for more general structures like simple or semisimple algebras. Recently, Tignol and Wadsworth introduced in [8] a new type of value function called *gauge*. For a value

function α on an algebra A over a field F with a valuation v , there is an *associated* graded algebra $\text{gr}_\alpha(A)$ over the graded field $\text{gr}_v(F)$. Basically, the conditions for α to be a gauge are: $\text{gr}_\alpha(A)$ is a semisimple graded algebra, and $\dim_{\text{gr}_v(F)} \text{gr}_\alpha(A) = \dim_F(A)$.

A significant feature of gauges is that they have good extension property. See [3, Theorem 4.3] for a proof of an existence theorem. But the proof given in [loc. cit.] is not constructive. So, it is worthwhile to obtain methods to construct gauges explicitly on algebras with well-known structure. In Theorem 3.2, we show how to construct a gauge in a quaternion algebra over a valued field. We also show that the graded algebra associated to the gauge constructed in Theorem 3.2 is a graded quaternion algebra. This result was motivated by the examples given in [8, Example 1.16].

2 Value functions and graded algebras

For the convenience of the reader, in this section, we review basic properties of value functions on finite-dimensional algebras from [8]. For a ring R , $R^\times$ denotes the group of units of R , $Z(R)$ the center of R , $J(R)$ the Jacobson radical of R , and $\mathbb{M}_k(R)$ the $k \times k$ matrix ring over R . We fix a totally ordered additive abelian group Γ , which contains the values of all valuations and the degrees of all gradings. Let ∞ be a symbol such that $\infty > \gamma$, and $\infty = \infty + \infty = \infty + \gamma = \gamma + \infty$, for all $\gamma \in \Gamma$.

A ring A is called a *graded ring* if there is a direct sum decomposition $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where each A_γ is an abelian group and $A_\gamma \cdot A_\delta \subseteq A_{\gamma+\delta}$, for all $\gamma, \delta \in \Gamma$. The *grade set* of A is the subset of Γ

$$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq \{0\}\}.$$

The elements in $\bigcup_{\gamma \in \Gamma} A_\gamma$ are the *homogeneous elements* of A . Every nonzero element $a \in A_\gamma$ is said to be homogeneous of degree γ , and is denoted by $\gamma = \deg(a)$. Every element $a \in A$ can be written in a unique way $a = \sum_{\gamma \in \Gamma} a_\gamma$, for $a_\gamma \in A_\gamma$ and a_γ is nonzero for a finite number of $\gamma \in \Gamma$. Each nonzero element a_γ on a decomposition of a is called a *homogeneous component* of a .

Let D be a division ring. A *valuation* on D is a map $v : D \rightarrow \Gamma \cup \{\infty\}$ such that for all $c, d \in D$

- (i) $v(d) = \infty$ if and only if $d = 0$;
- (ii) $v(cd) = v(c) + v(d)$;
- (iii) $v(c + d) \geq \min\{v(c), v(d)\}$.

Associated to a valuation v on D , we have a subring of D

$$R_v = \{d \in D \mid v(d) \geq 0\},$$

which is called *invariant valuation ring*, this is because it satisfies

- (V1) if $d \in D^\times$, then $d \in R_v$ or $d^{-1} \in R_v$;
- (V2) $dR_vd^{-1} = R_v$ for all $d \in D^\times$.

Conversely, if R is a subring of D satisfying (V1) and (V2), then there is a valuation v on D such that $R_v = R$. Moreover, $M_v = \{d \in D \mid v(d) > 0\}$ is the unique maximal left (and maximal right) ideal of R_v . So, M_v coincides with the Jacobson radical $J(R_v)$. The quotient R_v/M_v is

also a division ring called the *residue division ring* of v , and is denoted by $\overline{D}$. The *value group* of v is the subgroup $\Gamma_v = v(D^\times)$ of Γ .

If D is a field, then $\overline{D}$ is also a field, which is called a *residue field*. In this case, R_v is said just to be a *valuation ring*, because the condition (V2) above holds trivially. The main references for valuation theory on fields are the books [1] and [2]. For valuation theory on noncommutative division rings see [10] and the references given there.

A valuation v on a division ring D defines a filtration: for each $\gamma \in \Gamma$, we set

$$D^{\geq \gamma} = \{d \in D \mid v(d) \geq \gamma\}, \quad D^{> \gamma} = \{d \in D \mid v(d) > \gamma\} \quad \text{and} \quad D_\gamma = D^{\geq \gamma} / D^{> \gamma}.$$

Each $D^{\geq \gamma}$ is an additive abelian group, and $D^{> \gamma}$ is a subgroup of $D^{\geq \gamma}$. Then there is an abelian group

$$\text{gr}(D) = \bigoplus_{\gamma \in \Gamma} D_\gamma.$$

The multiplication is defined on each component by

$$\begin{aligned} D_\gamma \times D_\delta &\rightarrow D_{\gamma+\delta} \\ (c + D^{> \gamma}, d + D^{> \delta}) &\mapsto cd + D^{> \gamma+\delta}, \end{aligned}$$

and extended distributively for all elements of $\text{gr}_v(D)$, that makes $\text{gr}_v(D)$ into a graded ring called the *associated graded ring* of v on D . In this case, the grade set $\Gamma_{\text{gr}(D)}$ coincides with the value group Γ_v , and $D_0 = \overline{D}$, which is a division ring. For each $d \in D^\times$, write d' for the image of d in $\text{gr}(D)$, i.e., $d' = d + D^{> v(d)} \in D_{v(d)}$. Since $(d')^{-1} = (d^{-1})'$, we have $\text{gr}(D)$ is a *graded division ring*, i.e., every nonzero homogeneous element has an inverse. If D is commutative, then $\text{gr}(D)$ is also commutative and is then called a *graded field*.

Example 2.1 Let F be a field with a discrete valuation v , i.e., the value group Γ_v is isomorphic to the additive group of integers. So, we can assume $\Gamma_v = \mathbb{Z}$. Let $t \in F$ with $v(t) = 1$. For each homogeneous element $a \in \text{gr}_v(F)$ with $\deg(a) = n$, we have $a = (a(t')^{-n})(t')^n \in F_0 \cdot (t')^n$, where F_0 is the residue field $F^{\geq 0} / F^{> 0}$. Hence, $\text{gr}_v(F) = \bigoplus_{n \in \mathbb{Z}} F_0 \cdot (t')^n$, which is the Laurent polynomial ring $F_0[t', (t')^{-1}]$. To obtain a specific example, let p be a prime number, and let v_p be the p -adic valuation on the rational number field $\mathbb{Q}$, which is defined as follows:

$$v_p(0) = \infty \quad \text{and} \quad v_p\left(p^k \frac{a}{b}\right) = k,$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ are not divisible by p . In this case, the valuation ring of v_p is the localization $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$, with maximal ideal $J(\mathbb{Z}_{(p)}) = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b \text{ and } p|a\}$. So, the residue field $\mathbb{Z}_{(p)} / J(\mathbb{Z}_{(p)})$ is the finite field with p elements $\mathbb{F}_p$. So, the graded field $\text{gr}(\mathbb{Q})$ is the Laurent polynomial ring over a finite field.

Let M be a left D -vector space. A v -value function on M is a map $\alpha : M \rightarrow \Gamma \cup \{\infty\}$ such that

- (i) $\alpha(x) = \infty$ if and only if $x = 0$;
- (ii) $\alpha(x + y) \geq \min(\alpha(x), \alpha(y))$ for $x, y \in M$;
- (iii) $\alpha(dx) = v(d) + \alpha(x)$ for all $x \in M$ and $d \in D$.

A v -value function α is called a v -norm if M has finite dimension and contains a base $(x_i)_{i=1}^n$ such that

$$\alpha\left(\sum_{i=1}^n d_i x_i\right) = \min_{1 \leq i \leq n} (v(d_i) + \alpha(x_i)) \quad \text{for } d_1, \dots, d_n \in D.$$

Such a base $(x_i)_{i=1}^n$ is called a *splitting base* of M for α .

Given such an α on M , we can define an *associated graded module*: for each $\gamma \in \Gamma$, we set

$$M^{\geq \gamma} = \{x \in M \mid \alpha(x) \geq \gamma\}, \quad M^{> \gamma} = \{x \in M \mid \alpha(x) > \gamma\}, \quad \text{and } M_\gamma = M^{\geq \gamma} / M^{> \gamma}.$$

The associated graded structure

$$\text{gr}_\alpha(M) = \bigoplus_{\gamma \in \Gamma} M_\gamma$$

is a graded left $\text{gr}(D)$ -module such that $D_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$. Since $\text{gr}(D)$ is a graded division ring, $\text{gr}_\alpha(M)$ is a free module with a base of homogeneous elements. So, we call $\text{gr}_\alpha(M)$ a graded left $\text{gr}(D)$ -vector space, and we write $\dim_{\text{gr}(D)} \text{gr}_\alpha(M)$ for the rank of $\text{gr}(M)$ as a $\text{gr}(D)$ -module. Every nonzero element $x \in M$ has an image x' in $\text{gr}_\alpha(M)$, defined by $x' = x + M^{> \alpha(x)} \in M_{\alpha(x)}$. A D -base $(x_i)_{i=1}^n$ of M is a splitting base of M for α if and only if $(x'_i)_{i=1}^n$ is a $\text{gr}(D)$ -base of $\text{gr}_\alpha(M)$ (see [7, Corollary 2.3]). Hence, α is a v -norm on M if and only if $\dim_{\text{gr}(D)} \text{gr}_\alpha(M) = \dim_D M$.

Let F be a field with a valuation v , and let A be an F -algebra. A v -value function α on A is called *surmultiplicative* if

- (i) $\alpha(1) = 0$;
- (ii) $\alpha(xy) \geq \alpha(x) + \alpha(y)$, for all $x, y \in A$.

In this case, the multiplication on A induces a multiplication on the graded $\text{gr}(F)$ -vector space $\text{gr}_\alpha(A)$ such that for all $x, y \in A$

$$x'y' = xy + A^{> \alpha(x) + \alpha(y)} = \begin{cases} (xy)' & \text{if } \alpha(xy) = \alpha(x) + \alpha(y), \\ 0 & \text{if } \alpha(xy) > \alpha(x) + \alpha(y), \end{cases}$$

that makes $\text{gr}_\alpha(A)$ into a graded $\text{gr}(F)$ -algebra.

Simple or semisimple graded algebras can be defined analogously to the ungraded case. Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a graded algebra over a graded field F . A (left, right, or two-sided) ideal J of A is said to be *homogeneous* if $J = \bigoplus_{\gamma \in \Gamma} (J \cap A_\gamma)$, or equivalently, if $a = \sum_{\gamma \in \Gamma} a_\gamma \in J$, then each homogeneous component $a_\gamma \in J$. The graded algebra A is said to be *graded simple* if the only homogeneous two-sided ideals of A are $\{0\}$ and A . We say A is *graded semisimple* if A is a direct product of finitely many simple graded F -algebras. A good reference for the theory of simple graded algebras is the paper [4].

Definition 2.2 Let F be a field with a valuation v , and let A be a finite-dimensional F -algebra. A surmultiplicative v -value function α on A is called a v -gauge if α is a v -norm, i.e., $\dim_{\text{gr}(F)} \text{gr}_\alpha(A) = \dim_F(A)$, and $\text{gr}_\alpha(A)$ is a semisimple graded $\text{gr}(F)$ -algebra.

Remark 2.3 If A carries a gauge α then A must be semisimple because $\text{gr}_\alpha(A)$ is graded semisimple (see [8, Definition 1.4]). The degree zero part A_0 of $\text{gr}_\alpha(A)$ is also semisimple, even when $\text{gr}_\alpha(A)$ is simple (see [8, Proposition 2.1]).

Example 2.4 Let D be a finite-dimensional division algebra over its center $Z(D) = F$. Suppose D has a valuation w extending a valuation of the center v of F , i.e., $w|_F = v$. It is clear from the definition of valuation that w is a surmultiplicative v -value function. Since $\text{gr}(D)$ is a graded division algebra, it is graded simple, so semisimple. By [4, p. 79],

$$\dim_{\text{gr}(F)} \text{gr}(D) = \dim_F D \cdot |\Gamma_w : \Gamma_v|.$$

Hence, w is a v -gauge if and only if

$$\dim_F D = \dim_{\overline{F}} \overline{D} \cdot |\Gamma_w : \Gamma_v|. \quad (1)$$

The equality (1) means the valuation w is *defectless* (see for instance [10, p. 393]). Therefore, a valuation is not always a gauge, but every defectless valuation is a gauge. The equality (1) holds whenever $\text{char}(\overline{F}) \nmid [D : F]$ or v is a discrete valuation. Examples of division algebras with defect valuation are given in [5].

Remark 2.5 The defectless condition on division algebras (1) has been generalized for finite-dimensional semisimple algebras in [3, §1.b.]. Moreover, this is a necessary and sufficient condition for the existence of gauges (see [3, Proposition 1.9 and Theorem 4.3]).

3 Value functions on quaternion algebras

Let F be a field with $\text{char}(F) \neq 2$, and let $a, b \in F^\times$. A *quaternion algebra* $(a, b)_F$ is the F -algebra generated by $\{i, j\}$, such that

$$i^2 = a, \quad j^2 = b, \quad \text{and} \quad ij = -ji. \quad (2)$$

Set $k = ij$. It is easy to check that $(a, b)_F$ is a four-dimensional F -algebra with base $\{1, i, j, k\}$. Moreover, it is well-known (see for instance [6, Lemma 1.6]) that $(a, b)_F$ is a simple F -algebra with $Z((a, b)_F) = F$. Conversely, every four dimensional simple F -algebra A with $Z(A) = F$ is isomorphic to a quaternion algebra (see [6, Theorem 13.1]).

A graded quaternion algebra can be defined analogously to the ungraded case. Let F be a graded field with $\text{char}(F_0) \neq 2$. Let a and b be nonzero homogeneous elements of F . Let $(a, b)_F^{\text{gr}}$ be the F -algebra generated by i, j satisfying the relations (2). Set $k = ij$. We define a grading on $(a, b)_F^{\text{gr}}$ by setting

$$\deg(i) = \frac{1}{2}\deg(a), \quad \deg(j) = \frac{1}{2}\deg(b), \quad \text{and} \quad \deg(k) = \frac{1}{2}\deg(a) + \frac{1}{2}\deg(b).$$

Hence, $(a, b)_F^{\text{gr}}$ is a graded F -algebra having $\{1, i, j, k\}$ as a base of homogeneous elements.

Proposition 3.1 $(a, b)_F^{\text{gr}}$ is a simple graded F -algebra with $Z((a, b)_F^{\text{gr}}) = F$.

Proof: Let J be a nonzero homogeneous two-sided ideal of $(a, b)_F^{\text{gr}}$, and let $u = x + yi + zj + wk$ be a nonzero element of J . Then, at least one of x, y, z, w is nonzero. Suppose $y \neq 0$. The proof of the other cases is analogous. Since J is a two-sided ideal,

$$(a^{-1}i)ui = x + yi - zj - wk \in J.$$

Hence, $x + yi = 2^{-1}(u + (a^{-1}i)ui) \in J$. Likewise,

$$(b^{-1}j)(x + yi)j = x - yi \in J,$$

which implies that $yi \in J$. Hence, $y = (a^{-1}i)(yi) \in J$. Now let y_γ be a homogeneous component of y . Since J is a homogeneous ideal, $y_\gamma \in J$. But y_γ is a homogeneous element of F , which is a graded field. So, $1 = y_\gamma y_\gamma^{-1} \in J$. It remains to check that $Z((a, b)_F^{gr}) = F$. The inclusion $F \subseteq Z((a, b)_F^{gr})$ is clear. For the reverse inclusion, let $u = x + yi + zj + wk \in Z((a, b)_F^{gr})$. Since $iu = ui$, it follows that

$$xi + ay + zk + waj = xi + ay - zk - waj.$$

Thus, $z = w = 0$, because $\text{char}(F_0) \neq 2$ and every graded field is a domain with the grading forgotten. Likewise, it follows from $ju = uj$ that $xj - yk = xj + yk$. Hence, $y = 0$. Therefore, $u \in F$. ■

The next theorem is the main result of this paper.

Theorem 3.2 *Let F be a field with $\text{char}(\overline{F}) \neq 2$. Let v be a valuation on F such that $\text{char}(F) \neq 2$. Let $A = (a, b)_F$ be a quaternion F -algebra. Then $\alpha : A \rightarrow \Gamma \cup \{\infty\}$, defined by*

$$\alpha(x + yi + zj + wk) = \min(v(x), v(y) + \frac{1}{2}v(a), v(z) + \frac{1}{2}v(b), v(w) + \frac{1}{2}v(a) + \frac{1}{2}v(b)),$$

is a v -gauge on A . Moreover, the associated graded $\text{gr}(F)$ -algebra $\text{gr}_\alpha(A)$ is a graded quaternion algebra.

Proof: By definition, α is a v -norm and $\{1, i, j, k\}$ is a splitting base of A for α . We proceed to show that α is surmultiplicative. Clearly, $\alpha(1) = 0$. Note that

$$\alpha(ij) = \alpha(k) = \frac{1}{2}v(a) + \frac{1}{2}v(b) = \alpha(i) + \alpha(j) = \alpha(ji). \quad (3)$$

Likewise,

$$\begin{aligned} \alpha(ik) &= \alpha(i) + \alpha(k) = \alpha(ki), & \alpha(jk) &= \alpha(j) + \alpha(k) = \alpha(kj), \\ \alpha(i^2) &= 2\alpha(i), & \alpha(j^2) &= 2\alpha(j), & \text{and} & \alpha(k^2) = 2\alpha(k). \end{aligned} \quad (4)$$

To simplify notation, set $u_1 = 1$, $u_2 = i$, $u_3 = j$, and $u_4 = k$. By (3) and (4),

$$\alpha(u_i u_j) = \alpha(u_i) + \alpha(u_j) \quad \text{for } 1 \leq i, j \leq 4.$$

Let $x = \sum_{i=1}^4 x_i u_i$ and $y = \sum_{j=1}^4 y_j u_j$ in A , with $x_i, y_j \in F$. Then,

$$\begin{aligned} \alpha(xy) &= \alpha\left(\sum_{i,j=1}^4 x_i y_j u_i u_j\right) \geq \min_{1 \leq i, j \leq 4} (\alpha(x_i y_j u_i u_j)) \\ &= \min_{1 \leq i, j \leq 4} (v(x_i) + v(y_j) + \alpha(u_i) + \alpha(u_j)) \\ &\geq \min_{1 \leq i \leq 4} (v(x_i) + \alpha(u_i)) + \min_{1 \leq j \leq 4} (v(y_j) + \alpha(u_j)) \\ &= \alpha(x) + \alpha(y). \end{aligned}$$

Hence, α is a surmultiplicative v -value function. Since $\{1, i, j, k\}$ is a splitting base of A for α , we have $\{1', i', j', k'\}$ is a graded $\text{gr}(F)$ -base of $\text{gr}_\alpha(A)$. It follows from formulas (3) and (4) that

$$(i')^2 = (i^2)' = a', \quad (j')^2 = (j^2)' = b', \quad \text{and} \quad i'j' = k' = -j'i'.$$

Hence, $\text{gr}_\alpha(A)$ is the graded quaternion algebra $(a', b')_{\text{gr}(F)}^{\text{gr}}$, which is graded simple by Proposition 3.1. Therefore, α is a v -gauge. ■

Example 3.3 Consider the quaternion division algebra $(-1, -c)_{\mathbb{Q}}$ over the rational number field $\mathbb{Q}$ with base $\{1, i, j, k\}$, for a positive integer c . We have $\mathbb{Q}(i)$ is a quadratic extension of $\mathbb{Q}$ in $(-1, -c)_{\mathbb{Q}}$. Since $5 = (2 - i)(2 + i)$ in $\mathbb{Z}_{(5)}[i]$, the 5-adic valuation v_5 in $\mathbb{Q}$ has two distinct extensions to $\mathbb{Q}(i)$. Hence, v_5 does not extend to a valuation on the quaternion algebra $(-1, -c)_{\mathbb{Q}}$ by [9]. Suppose $5 \nmid c$. We can apply Theorem 3.2 to construct a v_5 -gauge on $(-1, -c)_{\mathbb{Q}}$ defined by

$$\alpha(x + yi + zj + wk) = \min(v(x), v(y), v(z), v(w)).$$

As in Example 2.1, $\text{gr}(\mathbb{Q})$ is the Laurent polynomial ring $\mathbb{F}_5[t, t^{-1}]$, where $\mathbb{F}_5$ is the finite field with five elements. Since $\deg(i) = \deg(j) = \deg(k) = 0$, the degree zero part of $\text{gr}_\alpha((-1, -c)_{\mathbb{Q}})$ is the quaternion algebra $(-1, -c)_{\mathbb{F}_5}$. But $(-1, -c)_{\mathbb{F}_5}$ is isomorphic to the matrix algebra $\mathbb{M}_2(\mathbb{F}_5)$, because -1 is a square in $\mathbb{F}_5$ or by Wedderburn's Finite Division Algebra Theorem (see [6, p. 248]). Hence, $\text{gr}_\alpha((-1, -c)_{\mathbb{Q}})$ is graded simple, but not a graded division algebra.

Remark 3.4 If there is a valuation on a division algebra extending a valuation of the center, then it is the unique extension (see [9]). On the other hand, the gauge constructed in Theorem 3.2 is not the only gauge that can be constructed on a quaternion algebra. See [3, Example 5.1] for an example of a quaternion algebra with infinitely many gauges extending a valuation of the center.

Referências

- [1] Efrat, I. *Valuations, orderings, and Milnor K-theory*. Mathematical Surveys and Monographs, **124**. American Mathematical Society, Providence, RI, 2006.
- [2] Engler, A. J.; Prestel, A. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.
- [3] Ferreira, M. A.; Wadsworth A. R. Value functions and Dubrovin valuation rings on simple algebras, *Trans. Amer. Math. Soc.*, to appear; preprint available at [arXiv:1307.5501](https://arxiv.org/abs/1307.5501).
- [4] Hwang, Y.-S.; Wadsworth, A. R. Correspondences between valued division algebras and graded division algebras. *J. Algebra*. **220** (1999), no. 1, 73–114.
- [5] Morandi, P. J., On defective division algebras, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), eds. B. Jacob and A. Rosenberg, pp. 359–367, *Proc. Sympos. Pure Math.*, **58**, Part 2, Amer. Math. Soc., Providence, RI, 1995.
- [6] Pierce, R. S. *Associative algebras*. Graduate Texts in Mathematics, 88. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982.

-
- [7] Renard, J.-F.; Tignol, J.-P.; Wadsworth, A. R. Graded Hermitian forms and Springer's theorem. *Indag. Math. (N.S.)* **18** (2007), no. 1, 97–134.
 - [8] Tignol, J.-P.; Wadsworth A. R. Value functions and associated graded rings for semisimple algebras, *Trans. Amer. Math. Soc.*, **362** (2010), 687–726.
 - [9] Wadsworth, A. R. Extending valuations to finite-dimensional division algebras. *Proc. Amer. Math. Soc.* 98 (1986), no. 1, 20–22.
 - [10] Wadsworth, A. R. Valuation theory on finite dimensional division algebras. *Valuation theory and its applications*, Vol. I (Saskatoon, SK, 1999), 385–449, Fields Inst. Commun., 32, Amer. Math. Soc., Providence, RI, 2002.