# A new non-conformable derivative based on Tsallis's $q$-exponential function

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## Uma nova derivada não-conforme baseada na função $q$-exponencial de Tsallis

**Resumo**

Neste artigo, uma nova derivada do tipo local é proposta e algumas propriedades básicas são estudadas. Esta nova derivada satisfaz algumas propriedades do cálculo de ordem inteira, por exemplo linearidade, regra do produto, regra do quociente e a regra da cadeia. Devido à função exponencial generalizada de Tsallis, podemos estender alguns dos resultados clássicos, a saber: teorema de Rolle, teorema do valor médio. Apresentamos a correspondente $Q$-integral a partir da qual surgem novos resultados. Especificamente, generalizamos a propriedade de inversão do teorema fundamental do cálculo e provamos um teorema associado à integração clássica por partes. Finalmente, apresentamos uma aplicação envolvendo equações diferenciais lineares por meio da $Q$-derivada.

**Palavras-chave:** Cálculo Fracionário; Derivada não compatível; Função $q$-exponencial.

**Abstract**

In this paper, a new derivative of local type is proposed and some basic properties are studied. This new derivative satisfies some properties of integer-order calculus, e.g. linearity, product rule, quotient rule and the chain rule. Because Tsallis' generalized exponential function, we can extend some of the classical results, namely: Rolle's theorem, the mean-value theorem. We present the corresponding $Q$-integral from which new results emerge. Specifically, we generalize the inversion property of the fundamental theorem of calculus and prove a theorem associated with the classical integration by parts. Finally, we present an application involving linear differential equations by means of $Q$-derivative.

**Keywords:** Fractional calculus; Non-conformable derivative; q-exponential function.

**MSC:** 26A33; 34K37.

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## 1 INTRODUCTION

The concept of derivatives of non-integer order first appeared, namely, on September $30^{\text{th}}$, 1695, in a letter between L'Hopital and Leibniz in which the question of a half-order derivative was posed Ross [1]. Then many scientists have tried to establish this theory in order to give a meaning to non-integer derivatives. Actually, there are more than 30 definitions were proposed under the title of "fractional derivatives" [2]. Furthermore, important notes about some of the new concepts are given, for example, [3, 4, 5] and some of the references therein.

Khalil *et al.* [6] proposed an extension of the ordinary limit definition for the derivative of a function, namely, the conformable derivative. Nevertheless, some authors have proposed other derivatives under the same approach: $\beta$-derivate [7], $M$-derivative [8], $G_T^\alpha$ derivative [9]. More recently, a non-conformable local derivative is introduced in [10]. These results are especially remarkable, since they allow to open a new direction on conformable calculus.

Here, we will refer to such derivatives simply as conformable derivatives or local derivatives of arbitrary order, since these derivatives can be written in terms of the ordinary derivative of order one, so we will avoid the term "fractional".

These conformable operators have properties similar to the classical calculus. Abdeljawad [11] proves versions of chain rules, Gronwall's inequality, integration by parts, Taylor power series expansions and Laplace transforms for the conformable calculus. The conformable operator has recently occurred in many scientific fields [12, 13, 14, 15]. For example, a successful implemention is to apply the conformable derivative to non-Darcian flow in low-permeability porous media [16].

Deformations of the exponential functions are considered in three main complementary directions: formal mathematical developments; observation of consistent concordance with experimental behavior; and theoretical physical developments [17]. Many other $q$-representations were proposed such as the $q$-algebra [18], the $q$-Gaussian [19] and so on.

The main purpose of this study is to propose a new definition of fractional derivative that generalizes the fractional derivatives To do this, Section 2 is devoted to the q-exponential function and some fundamental properties. The new derivatives and fractional integrals re given in Sections 3 and 4. In Section 5, an application is presented and Section 6 draws some conclusions.

## 2 $Q$-EXPONENTIAL

Fix a number $q \geq 0$. The deformed exponential function, called the Tsallis $q$-exponential function [18, 20], can be introduced as solution of the first order differential equation

$$y'(x) = y^q(x), \text{ with } y(0) = 1. \tag{1}$$

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Solution of (1) is

$$
e_q^x = \begin{cases} (1 + (1-q)x)^{\frac{1}{1-q}}, & \text{for } q \neq 1 \text{ and } 1 + (1-q)x \geq 0, \\ e^x, & \text{for } q = 1, \\ 1 + x, & \text{for } q = 0. \end{cases} \tag{2}
$$

From the relation of the Eq. (2), concerning the domain and codomain of the $q$-exponential function, we observe

$$
\begin{aligned}
q = 0 &: \quad e_0 : \mathbb{R} \longrightarrow \mathbb{R}, \\
0 < q < 1 &: \quad e_q : \left[\frac{1}{q-1}, \infty\right) \longrightarrow [0, \infty), \\
q = 1 &: \quad e_1 : \mathbb{R} \longrightarrow (0, \infty), \\
q > 1 &: \quad e_q : \left(-\infty, \frac{1}{q-1}\right] \longrightarrow (0, \infty).
\end{aligned} \tag{3}
$$

**Remark 2.1.** *In general,* $\left(e_q^x\right)^a \neq e_q^{ax}$, *except for* $q = 1$.

**Theorem 2.2** ([19])*. For* $0 < q \leq 1$, $x \in [0, \infty)$, $e_q^x$ *is positive, continuous, increasing, convex and such that* $\lim_{x \to \infty} e_q^x = \infty$.

The interest of this topic comes from the fact that $u(x) = C\left[1 + (1-q)\, C^{q-1}\, x\right]^{\frac{1}{1-q}}$ is the unique solution to the initial value problem

$$
\begin{cases} u'(x) = u^q(x), \\ u(0) = C > 0. \end{cases} \tag{4}
$$

Scaling analysis can identify whether a problem, say for $u = u(x)$, will admit a similarity solution by making use of the change of variables

$$
u(x) = \mathbf{U} \cdot \tilde{u}(\tilde{x}), \quad x = \mathbf{L} \cdot \tilde{x}, \tag{5}
$$

where $\mathbf{U}, \mathbf{L}$ are undetermined real positive parameters. We call the problem scale invariant if relationships exist between the scaling parameters $\mathbf{U}, \mathbf{L}$ in (5) that make the scaled problem take exactly the same form as the original problem with at least one scaling parameter remaining undetermined,

$$
\text{Problem for } u(x) \iff \text{Problem for } \tilde{u}(\tilde{x}). \tag{6}
$$

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Substituting the change of variables (5) in Eq. (4), we obtain

$$
\begin{cases}
\mathbf{U} \cdot \dfrac{d\tilde{u}}{d\tilde{x}} = \mathbf{L}\mathbf{U}^q \cdot \tilde{u}^q(\tilde{x}), \\
\mathbf{U} \cdot \tilde{u}(0) = C.
\end{cases}
\tag{7}
$$

Setting $\mathbf{L}\mathbf{U}^{q-1} = 1$ and $\mathbf{U} = C > 0$ eliminates the scaling constants from (7) and makes the scaled system identical with (4) and hence the problem is scale invariant. From these two relationships, we can express $\mathbf{L}$ in term of the free parameter $\mathbf{U}$,

$$
\mathbf{L} = \mathbf{U}^{q-1}.
\tag{8}
$$

Consequently, if we have one solution of (4), say $u(x) = \tilde{u}(\tilde{x})$, then other solutions are given by the one-parameter continuous family,

$$
u(x) = C\tilde{u}\left( \frac{x}{C^{1-q}} \right),
\tag{9}
$$

for arbitrary values of $C$. In other words, note that (4) is invariant under the transformation

$$
\tilde{u} = \frac{u}{C}, \quad \tilde{x} = \frac{x}{C^{1-q}}.
\tag{10}
$$

By direct inspection it is verified that the function $\tilde{u}(\tilde{x}) = e_q^{\tilde{x}}$ satisfies the first order differential equation

$$
\frac{d\tilde{u}}{d\tilde{x}} = \tilde{u}^q(\tilde{x})
\tag{11}
$$

subject to the initial condition $\tilde{u}(0) = 1$. For more details, see [17] and the references therein.

## 3   LOCAL $Q$-DERIVATIVE

In this section, we give our new definition of a non-conformable fractional derivative of a function in a point $x$ and obtain classical generalized properties of integer-order calculus.

Let us begin with the following definition, which is a generalization of the usual definition of a derivative as a special limit.

**Definition 3.1.** *Given a function $f : [0, \infty) \to \mathbb{R}$. Then, the non-conformable fractional derivative $\mathcal{D}_Q^{\alpha;q}(f)(x)$ of order $\alpha$ of $f$ at $x$ is defined by*

$$
\mathcal{D}_Q^{\alpha,q} f(x) = \lim_{\varepsilon \to 0} \frac{f\left( x + \varepsilon e_q^{x^{-\alpha}} \right) - f(x)}{\varepsilon},
\tag{12}
$$

*for all $x > 0$, $0 < \alpha \le 1$, $0 < q \le 1$.*

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*For $x = 0$, we adopt the following definition:*

$$\mathcal{D}_Q^{\alpha,q} f\left(0\right) = \lim_{x \to 0^+} \mathcal{D}_Q^{\alpha,q} f\left(x\right). \tag{13}$$

*In addition, if the non-conformable fractional derivative $\mathcal{D}_Q^{\alpha,q}$ of $f$ of order $\alpha$ exists, then we simply say it is $Q$-differentiable.*

**Definition 3.2** ([21]). *Let $\alpha, q \in (0, 1]$. A differential operator $\mathcal{D}_Q^{\alpha,q}$ is conformable if and only if $\lim_{\alpha \to 0} \mathcal{D}_Q^{\alpha,q}$ is an identity operator and $\mathcal{D}_Q^{1,q}$ is the classical differential operator.*

**Remark 3.3.** *The derivative $\mathcal{D}_Q^{\alpha,q} f$ is not a fractional (order) derivative. It is exactly the integer-order derivative times the root function $e_q^{x^{-\alpha}}$, therefore, any comparison with the classical fractional derivatives is erroneous, because we are considering mathematical objects of different kinds [3, 4, 5, 9, 10].*

**Theorem 3.4.** *Fix $\alpha, q \in (0, 1]$ and $x > 0$. A function $f : [0, \infty[ \to \mathbb{R}$ is $Q$-differentiable at $x$ if and only if $f$ is differentiable at $x$; in this case, we have*

$$\mathcal{D}_Q^{\alpha,q} f(x) = e_q^{x^{-\alpha}} f'\left(x\right). \tag{14}$$

*Proof.* Suppose that $\mathcal{D}_Q^{\alpha,q} f(x)$ exists then

$$\mathcal{D}_Q^{\alpha,q} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon e_q^{x^{-\alpha}}) - f(x)}{\varepsilon} = e_q^{x^{-\alpha}} \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{15}$$

with $h = \varepsilon e_q^{x^{-\alpha}}$. Conversely, using Definition (3.1) and in view of L'Hôpital's rule, we get

$$\lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon e_q^{x^{-\alpha}}\right) - f\left(x\right)}{\varepsilon} = \lim_{\varepsilon \to 0} e_q^{x^{-\alpha}} f'\left(x + \varepsilon e_q^{x^{-\alpha}}\right) = e_q^{x^{-\alpha}} f'\left(x\right). \tag{16}$$

The proof is complete. $\qquad\square$

Hence, $\mathcal{D}_Q^{\alpha,q} f(x) = e_q^{x^{-\alpha}} f'\left(x\right)$. Of course, for $x = 0$ this is not valid and it would be useful to deal with equations and solutions with singularities. In particular, if $q = 1$, then we obtain the non-conformable derivative defined by Guzman [10].

According to Definition (3.1) and Theorem (3.4), the non-conformable derivatives of some functions are given as shown in Theorem (3.5).

**Theorem 3.5.** *Let $a, n \in \mathbb{R}$, $\alpha, q \in (0, 1]$, and suppose $f$ is $Q$-differentiable at point $x > 0$, then*

(a) $\mathcal{D}_Q^{\alpha,q} 1 = 0$.

(b) $\mathcal{D}_Q^{\alpha,q} x^n = n e_q^{x^{-\alpha}} x^{n-1}$.

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*(b)* $\mathcal{D}_Q^{\alpha;q} e^{ax} = a e_q^{x^{-\alpha}} e^{ax}$.

*(c)* $\mathcal{D}_Q^{\alpha;q} \sin ax = a e_q^{x^{-\alpha}} \cos ax$.

*(d)* $\mathcal{D}_Q^{\alpha;q} \cos ax = -a e_q^{x^{-\alpha}} \sin ax$.

Another consequence of the Definition (3.1), we obtain the following result known in classical calculus.

**Theorem 3.6.** *Let $\alpha, q \in (0,1]$, $a, b \in \mathbb{R}$ and $f, g$ Q-differentiable at the point $x > 0$. Then:*

*(a)* $\mathcal{D}_Q^{\alpha;q} (af + bg)(x) = a\mathcal{D}_Q^{\alpha;q} f(x) + b\mathcal{D}_Q^{\alpha;q} g(x)$.

*(b)* $\mathcal{D}_Q^{\alpha;q} (f \cdot g)(x) = f(x)\mathcal{D}_Q^{\alpha;q} g(x) + g(x)\mathcal{D}_Q^{\alpha;q} f(x)$.

*(c)* $\mathcal{D}_Q^{\alpha;q} \left(\dfrac{f}{g}\right)(x) = \dfrac{g(x)\mathcal{D}_Q^{\alpha;q} f(x) - f(x)\mathcal{D}_Q^{\alpha;q} g(x)}{[g(x)]^2}$.

*(d)* $\mathcal{D}_Q^{\alpha;q}(C) = 0$, *where $f(x) = C$ is a constant.*

*Proof.* If $f, g$ are $Q$-differentiable, then they are in fact differentiable, and we have $\mathcal{D}_Q^{\alpha;q} f(x) = e_q^{x^{-\alpha}} f'(x)$, and $\mathcal{D}_Q^{\alpha;q} g(x) = e_q^{x^{-\alpha}} g'(x)$. To prove item (a), since $f, g$ are differentiable, then so is $af + bg$. By Theorem (3.4), $af + bg$ is $Q$-differentiable and we have

$$\mathcal{D}_Q^{\alpha;q}(af+bg)(x) = e_q^{x^{-\alpha}} \frac{d}{dx}(af+bg)(x) = a e_q^{x^{-\alpha}} f'(x) + b e_q^{x^{-\alpha}} g'(x) = a\mathcal{D}_Q^{\alpha;q} f(x) + b\mathcal{D}_Q^{\alpha;q} g(x).$$
$$(17)$$

The proofs of items (b) through (d) are as trivial as the proof of (a). $\square$

**Theorem 3.7.** *The index law, that is, $\mathcal{D}_Q^{\alpha;q} \mathcal{D}_Q^{\beta;q} f(x) = \mathcal{D}_Q^{\alpha+\beta;q} f(x)$ for any $\alpha, \beta$ does not hold in general.*

In the works of Guzman [10] and Fleitas [24], we can see that the chain rule is valid for non-conformable derivatives.

**Theorem 3.8.** *Assume $f, g : [0, \infty) \to \mathbb{R}$ are Q-differentiable functions, where $\alpha, q \in (0,1]$. Then $f \circ g$ is Q-differentiable at $x > 0$, and $\mathcal{D}_Q^{\alpha;q}(f \circ g)(x) = f'(g(x))\,\mathcal{D}_Q^{\alpha;q} g(x)$.*

*Proof.* In fact, by Theorem (3.4), if $f, g$ are $Q$-differentiable on $(0, \infty)$, then they are in fact differentiable on $(0, \infty)$, and we have

$$\mathcal{D}_Q^{\alpha;q} h(x) = \mathcal{D}_Q^{\alpha;q}(f \circ g)(x) = e_q^{x^{-\alpha}} (f \circ g)'(x) = e_q^{x^{-\alpha}} f'(g(x)) g'(x) = f'(g(x))\mathcal{D}_Q^{\alpha;q} g(x). \quad (18)$$

$\square$

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**Theorem 3.9.** *Let $0 < \alpha, q \leq 1$, and $x > 0$. Then we have the following results:*

*(a)* $\mathcal{D}_Q^{\alpha,q} \left( e_q^{x^{-\alpha}} \right) = -\dfrac{\alpha}{x^{\alpha+1}} \left( e_q^{x^{-\alpha}} \right)^{q+1}.$

*(b)* $\mathcal{D}_Q^{\alpha,q} \sin \left( e_q^{x^{-\alpha}} \right) = -\dfrac{\alpha}{x^{\alpha+1}} \left( e_q^{x^{-\alpha}} \right)^{q+1} . \cos \left( e_q^{x^{-\alpha}} \right).$

*(c)* $\mathcal{D}_Q^{\alpha,q} \cos \left( e_q^{x^{-\alpha}} \right) = \dfrac{\alpha}{x^{\alpha+1}} \left( e_q^{x^{-\alpha}} \right)^{q+1} . \sin \left( e_q^{x^{-\alpha}} \right).$

*(d)* *If $f$ is $Q$-differentiable at $x$ and $f(x) \neq 0$, then* $\mathcal{D}_Q^{\alpha,q} \left( |f(x)| \right) = \dfrac{f(x) \mathcal{D}_Q^{\alpha,q} f(x)}{|f(x)|}.$

*Proof.* Let us start with (d). From Definition (3.1) we have

$$
\begin{aligned}
\mathcal{D}_Q^{\alpha,q} \left( |f(x)| \right) &= \lim_{\varepsilon \to 0} \frac{\left| f \left( x + \varepsilon e_q^{x^{-\alpha}} \right) \right| - |f(x)|}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left[ f \left( x + \varepsilon e_q^{x^{-\alpha}} \right) \right]^2 - [f(x)]^2}{\varepsilon \left( \left| f \left( x + \varepsilon e_q^{x^{-\alpha}} \right) \right| + |f(x)| \right)} \\
&= \lim_{\varepsilon \to 0} \frac{\left[ f \left( x + \varepsilon e_q^{x^{-\alpha}} \right) \right]^2 - [f(x)]^2}{\varepsilon} \cdot \frac{1}{\left( \left| f \left( x + \varepsilon e_q^{x^{-\alpha}} \right) \right| + |f(x)| \right)} \\
&= \mathcal{D}_Q^{\alpha,q}[f(x)]^2 \cdot \frac{1}{2|f(x)|} \\
&= 2f(x) \mathcal{D}_Q^{\alpha,q} f(x) \cdot \frac{1}{2|f(x)|},
\end{aligned}
\tag{19}
$$

which proves the intended relation. The items (1) through (3) follow directly from the Theorem (3.8). $\qquad\square$

We now prove the extension, for $Q$-differentiable functions in the sense of the local $Q$-derivative defined in Eq. (3.1), of Rolle's theorem.

**Theorem 3.10** (Mean value theorem for $Q$-differentiable functions)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a function such that:*

*(i) $f$ is continuous on $[a, b]$;*

*(ii) $f$ is $Q$-differentiable on $(a, b)$ for some $\alpha \in (0, 1)$.*

*Then, there exists $c \in (a, b)$ such that*

$$
\mathcal{D}_Q^{\alpha,q} f(c) = -\frac{f(b) - f(a)}{\dfrac{e_q^{b^{-\alpha}}}{\alpha} - \dfrac{e_q^{a^{-\alpha}}}{\alpha}} \frac{\left( e_q^{c^{-\alpha}} \right)^{q+1}}{c^{\alpha+1}},
$$

*with $\alpha, q \in (0, 1]$.*

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*Proof.* Once again, by Theorem (3.4), the condition (ii) implies that $f$ is differentiable on $(a, b)$ and $\mathcal{D}_Q^{\alpha, q} f(x) = e_q^{x^{-\alpha}} f'(x)$ on $(a, b)$. Now, apply the classical Cauchy mean value theorem to the functions $f(x)$ and $g(x) = -\frac{e_q^{x^{-\alpha}}}{\alpha}$ on the interval $(a, b)$. This completes the proof of the theorem. $\qquad\square$

As a consequence of the previous theorem, we have the following

**Theorem 3.11.** *Let $0 < \alpha \leq 1$, $0 < a < b$, and $F, G$ $Q$-differentiable at all $x \in (a, b)$. Then there exists a constant $C$ such that $F(x) = G(x) + C$, with $q \in (0, 1]$.*

*Proof.* Simply apply the above theorem to $H(x) = F(x) - G(x)$. $\qquad\square$

**Definition 3.12.** *Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, $0 < q \leq 1$ and $f$ $n$-differentiable for $x > 0$. Then the $Q$-differentiable of order $\alpha$ at $x > 0$ is given by*

$$\mathcal{D}_Q^{\alpha, q} f(x) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}\left(x + \varepsilon e_q^{x^{[\alpha]-\alpha-1}}\right) - f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{20}$$

*where $[\alpha]$ is the smallest integer greater than or equal to $\alpha$. For $x = 0$ we proceed in a similar way as in Definition (3.1).*

**Remark 3.13.**

- *As a consequence of Definition (3.12), one can easily show that $\mathcal{D}_Q^{\alpha, q} f(x) = e_q^{x^{[\alpha]-\alpha-1}} f^{([\alpha])}(x) = e_q^{x^{n-\alpha}} f^{(n+1)}(x)$, where $\alpha \in (n, n+1]$, and $f$ is $(n+1)$-differentiable at $x > 0$.*

- *Also, it is clear that this definition is generalization of Definition (3.1), putting $n = 0$ in (20) we obtain the condition (12), where $f^{(0)}(x) = f(x)$.*

## 4 $Q$-INTEGRAL

The analogous definition of the integral operator corresponding to the derivative operator is given by the following definition.

**Definition 4.1** ($Q$-integral). *Let $a \geq 0$ and $x \geq a$. Let $f$ be a function defined in $(a, t]$ and $0 < \alpha < 1$. Then, the $Q$-integral of order $\alpha$ of a function $f$ is defined by*

$$_Q\mathcal{I}_a^{\alpha, q} f(x) = \int_a^x f(s)\, d_{\alpha, q} s := \int_a^x \frac{f(s)}{e_q^{s^{-\alpha}}}\, ds, \tag{21}$$

*with $0 < q \leq 1$.*

**Theorem 4.2.** *Let $a \geq 0$ and $0 < \alpha \leq 1$. Also, let $f$ be a continuous function such that there exists $_Q\mathcal{I}_a^{\alpha, q} f$. Then*

$$\mathcal{D}_Q^{\alpha, q}(_Q\mathcal{I}_a^{\alpha, q} f(x)) = f(x), \tag{22}$$

*with $x \geq a$ and $0 < q \leq 1$.*

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*Proof.* Indeed, using the chain rule proved in Theorem (3.8) we have

$$
\begin{aligned}
\mathcal{D}_Q^{\alpha,q}\left({}_Q\mathcal{I}_a^{\alpha,\beta}f\left(x\right)\right) &= e_q^{x^{-\alpha}}\frac{d}{dx}({}_Q\mathcal{I}_a^{\alpha,q}f\left(x\right)) \\
&= e_q^{x^{-\alpha}}\frac{d}{dx}\left(\int_a^x \frac{f(s)}{e_q^{s^{-\alpha}}}\,ds\right) \\
&= e_q^{x^{-\alpha}}\frac{f(x)}{e_q^{x^{-\alpha}}} \\
&= f\left(x\right),
\end{aligned}
\tag{23}
$$

and the proof is complete. □

**Theorem 4.3** (Fundamental theorem of calculus)**.** *Let $f:(a,b)\to\mathbb{R}$ be an $Q$-differentiable function and $0<\alpha\leq 1$. Then, for all $t>a$ we have*

$$
{}_Q\mathcal{I}_a^{\alpha,q}\left(\mathcal{D}_Q^{\alpha,q}f\left(x\right)\right) = f\left(x\right) - f\left(a\right),
$$

*with $0<q\leq 1$.*

*Proof.* In fact, since function $f$ is differentiable, using the chain rule of Theorem (3.8) and the fundamental theorem of calculus for the integer-order derivative, we have

$$
\begin{aligned}
{}_Q\mathcal{I}_a^{\alpha,q}\left(\mathcal{D}_Q^{\alpha,q}f\left(x\right)\right) &= \int_a^x \mathcal{D}_Q^{\alpha,q}f\left(s\right)\,d_{\alpha,q}s \\
&= \int_a^x e_q^{s^{-\alpha}}\frac{d}{ds}f\left(s\right)\,d_{\alpha,q}s \\
&= \int_a^x \frac{df\left(s\right)}{ds}ds \\
&= f\left(x\right) - f\left(a\right)
\end{aligned}
\tag{24}
$$

□

If the condition $f(a)=0$ holds, then by Theorem (4.2), Eq. (24), we have ${}_Q\mathcal{I}_a^{\alpha,q}\left[\mathcal{D}_Q^{\alpha,q}f(x)\right] = f(x)$.

**Theorem 4.4.** *Let $\alpha,q\in(0,1]$, $0<a<b$ and $f\in L^1\left([a,b],\mathbb{R}\right)$. Then $\left|{}_Q\mathcal{I}_a^{\alpha,q}f(x)\right|\leq_Q \mathcal{I}_a^{\alpha,q}|f(x)|$.*

*Proof.* The result follows directly since

$$
\left|{}_Q\mathcal{I}_a^{\alpha,q}f(x)\right| = \left|\int_a^x \frac{f(s)}{e_q^{s^{-\alpha}}}\,ds\right| \leq \int_a^x \left|\frac{f(s)}{e_q^{s^{-\alpha}}}\right|\,ds \leq \int_a^x \frac{|f(s)|}{e_q^{s^{-\alpha}}}\,ds.
\tag{25}
$$

Therefore, $\left|{}_Q\mathcal{I}_a^{\alpha,q}f(x)\right| \leq_Q \mathcal{I}_a^{\alpha,q}|f(x)|$ and the proposition is proved. □

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**Theorem 4.5.** *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that $f, g$ are differentiable and $0 < \alpha < 1$. Then*

$$\int_a^b f(x) \mathcal{D}_Q^{\alpha;q} g(x) \, d_\alpha x = f(x) g(x) \big|_a^b - \int_a^b g(x) \mathcal{D}_Q^{\alpha;q} f(x) \, d_\alpha x, \tag{26}$$

*with $0 < q \leq 1$.*

*Proof.* Indeed, using the definition of $Q$-integral and applying the chain rule of Theorem (3.8) and the fundamental theorem of calculus for integer-order derivatives, we have

$$
\begin{aligned}
\int_a^b f(x) \mathcal{D}_Q^{\alpha;q} g(x) \, d_{\alpha,q} x &= \int_a^b f(x) \mathcal{D}_Q^{\alpha;q} g(x) \frac{dx}{e_q^{x-\alpha}} \\
&= \int_a^b f(x) g'(x) \, dx \\
&= f(x) g(x) \big|_a^b - \int_a^b g(x) e_q^{x-\alpha} \frac{df(x)}{dt} \frac{dx}{e_q^{x-\alpha}} \\
&= f(x) g(x) \big|_a^b - \int_a^b g(x) \mathcal{D}_Q^{\alpha;q} f(x) \, d_\alpha x.
\end{aligned}
$$

$\square$

# 5   APPLICATION IN NON-CONFORMABLE DIFFERENTIAL EQUATIONS

In this section, we will give a simple scheme for solving the non-conformable differential equations. We consider an equation in the form

$$\mathcal{D}_Q^{\alpha,q} u(x) + P(x) u(x) = Q(x). \tag{27}$$

where $P(x), Q(x)$ are $Q$-differentiable functions and $u(x)$ is unknown. According to Theorem (3.4), Eq. (27) is transformed to

$$\frac{d}{dx} u(x) + \frac{P(x)}{e_q^{x-\alpha}} u(x) = \frac{Q(x)}{e_q^{x-\alpha}}, \tag{28}$$

which is a first-order differential equation and easy to solve. By definition of $Q$-integral, we conclude that the solution is given by

$$u(x) = e^{-\int f(x) \, d_{\alpha,q} x} \left( \int \left( Q(s) e^{\int f(x) \, d_{\alpha,q} x} \right) d_{\alpha,q} x + C \right),$$

where $C$ is an arbitrary constant.

One can also find the fixed point of operator $\mathcal{D}_Q^{\alpha;q}$.

**Example 5.1.** $\mathcal{D}_Q^{\alpha;q} y(x) - y(x) = 0$, $y(0) = C > 0$, $0 < \alpha, q < 1$.

According to Theorem (3.4), $\mathcal{D}_Q^{\alpha;q} y(x) = y(x)$ is equivalent to $e_q^{x-\alpha} y'(x) = y(x)$. To

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solve this simple ODE, one can get the result: $y(x) = Ce^{\int \frac{1}{e_q^{x-\alpha}} \, dx}$, where $C$ is any positive constant.

Note that to integrate

$$\int (1 + (1-q)x^{-\alpha})^{\frac{1}{q-1}} \, dx, \tag{29}$$

we recall Chebyshev's theorem [22]: For rational numbers $m, n, r$ ($r \neq 0$) and nonzero real numbers $a, b$, the integral

$$\int x^m (a + bx^r)^n \, dx \tag{30}$$

is expressible by of the elementary functions if and only if at least one of the quantities

$$\frac{m+1}{r}, \quad n, \quad \frac{m+1}{r+n}, \tag{31}$$

is an integer.

In fact, the integral (30) may be rewritten as [23]

$$
\begin{aligned}
\int x^m (a + bx^r)^n \, dx &= \frac{1}{r} a^{\frac{p+1}{r}+n} b^{-\frac{m+1}{r}} B_y \left( \frac{1+m}{r}, n-1 \right) \\
&= \frac{1}{m+1} a^{\frac{m+1}{r}+n} b^{-\frac{m+1}{r}} y^{\frac{1+m}{r}} F \left( \frac{m+1}{r}, 2-n, \frac{1+m+r}{r}; y \right),
\end{aligned} \tag{32}
$$

where $y = \frac{b}{a}x^r$, and $B_y \left( \frac{1+m}{r}, q-1 \right)$ and $F \left( \frac{m+1}{r}, 2-q, \frac{1+m+r}{r}; y \right)$ are the incomplete beta function and hypergeometric function, respectively. Consequently Chebyshev's theorem indicates that the integration (29) is elementary if and only if either $-\frac{1}{\alpha}$ or $\frac{1}{q-1}$ or $\frac{1}{q-1} - \frac{1}{\alpha}$ is an integer number.

**Remark 5.2.** *Applying the scaling transformation* (10) *with* $q = 1$, *we have*

$$
\begin{cases}
\mathcal{D}_Q^{\alpha;q} \tilde{y}(\tilde{x}) = \tilde{y}(\tilde{x}), \\
\tilde{y}(0) = 1.
\end{cases} \tag{33}
$$

## 6 FINAL CONSIDERATIONS

In the present study, a definition for a new non-conformable derivative, called the $Q$-derivative, and its corresponding $Q$-integral, was proposed. We will also obtain several results for $\mathcal{D}_Q^{\alpha;q} f(x)$, which bear a great similarity with the results obtained in classical calculus. This result that the definition presented here can be considered consider a q-deformation of the non-conformable derivative introduced in Guzman [10]. This new generalized derivative will contribute to the solution of non-conformable differential equations. Another advantage of these operators is that they depend on two parameters naturally. We also believe that it will guide us to the next works.

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