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Article

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Robust Estimation for Discrete-Time Markovian Jump Linear Systems in a Data Fusion Scenario

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Estimação Robusta para Sistemas Lineares Sujeitos a Saltos Markovianos em Tempo Discreto em um Cenário de Fusão de Dados

Resumo

Este artigo considera o problema de estimação recursiva robusta para sistemas lineares sujeitos a saltos Markovianos de tempo discreto em um cenários de fusão de dados ponderados e probabilísticos. O problema é declarado em termos da otimização de um apropriado funcional quadrático em um cenário de fusão de dados. As estimativas aqui apresentadas foram desenvolvidas baseadas em sistemas com mais de uma equação de medida. Exemplos numéricos são apresentados para verificar a eficácia dos algoritmos propostos.

Palavras-chave: Sistemas Markovianos; Fusão de Dados; Robustez; Filtro de Kalman.

Abstract

This paper considers the problem of robust recursive estimation for discrete-time Markovian jump linear systems in both weighted and probabilistic data fusion scenarios. The problem is stated in terms of the optimization of an appropriate quadratic functional in a data fusion scenario. The estimates presented here were developed based on systems with more than one measurement equation. Numerical examples are presented to verify the effectiveness of proposed algorithms.

Keywords: Markovian Systems; Data Fusion; Robustness; Kalman Filter.

MSC (2020): 93E10; 93E11; 93E24.

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1 INTRODUCTION

The Discrete-Time Markovian Jump Linear Systems (DMJLS) are models of dynamical systems subject to abrupt changes in their parameters which are simulated by a Markov chain. The estimation theory for DMJLS has received great attention in literature as can be seen in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. An important paper in this field is the recursive approach proposed by [12] based on Riccati difference equation due to the augmented Markovian model defined to deal with the unknown Markovian chain. This approach is useful to be used in online applications.

An important point to note in the estimate's algorithm in [12] is that this was developed considering a single measurement equation. This makes it vulnerable in situations where this measurement equation may suffer measurement failure. Failure conditions or modelling errors in measurement are situations that must be considered in estimation processes, as they may disrupt the state estimation. In this context, data fusion approach can solve this problem, with systems that consider more than one measurement equation. Several papers have been proposed, in the literature, to solve these problems to DMJLS, see for instance [13, 14, 15, 16, 17, 18].

In this note, the robust estimation developed in [12] is extended for an approach in both weighted and probabilistic data fusion scenarios. These new estimates are useful when the DMJLS is subject to failure conditions or modelling errors. They were deduced through minimization of the worst-possible regularized residual norm based on data fusion approach proposed by [19]. Numerical examples were performed to show the effectiveness of proposed robust estimation.

This paper is organized as follows: in Section *DMJLS in Data Fusion Scenario*, we present the formulation for DMJLS in both weighted and probabilistic data fusion scenarios; in Section *Robust Estimation for DMJLS in Data Fusion Scenarios*, DMJLS robust estimates are deduced; in Section *Numerical Examples*, numerical examples are provided showing the effectiveness of robust filters proposed and in Section *Conclusion*, the conclusions are presented.

2 FORMULATION OF THE PROBLEM ESTIMATION FOR DMJLS IN DATA FUSION SCENARIO

In this section we present the formulation of the problem estimation for DMJLS in data fusion scenarios. The uncertain Markovian model in data fusion scenario is defined as an augmented model to deal with the unknown Markovian chain. We present both cases, a weighted scenario and a probabilistic scenario are considered.

2.1 Formulation of the Problem Estimation for DMJLS in Weighted Data Fusion

We present the problem estimation in weighted data fusion scenario. This problem consider the fusion of L measurement equations. Consider the uncertain measurement equations for

$k = 1, ..., L$

$$y_i^{(k)} = (H_{i,\Theta_i}^{(k)} + \delta H_{i,\Theta_i}^{(k)})x_i + D_{i,\Theta_i}^{(k)} v_i^{(k)};\ i \geq 0, \tag{1}$$

where $v_i^{(k)}$ are measurement noises and $\delta H_{i,\Theta_i}^{(k)}$ are uncertainties associated to parameter matrices $H_{i,\Theta_i}^{(k)}$. The unknown parameter vector x_i is the same for all measurements $y_i^{(k)}$. This situation describes several distorted measurements of an unknown vector x_i arising from different uncertain sources, as depicted in Fig. (1).

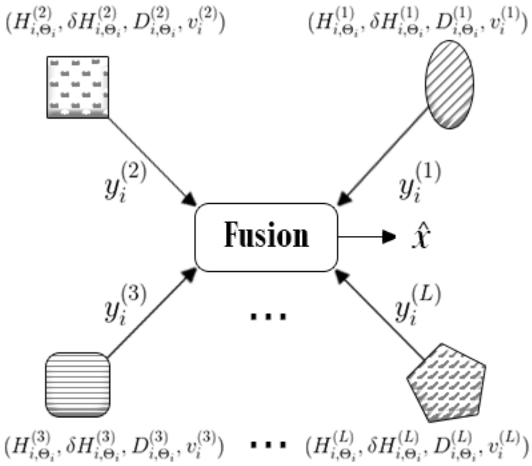

Figure 1: Data Fusion Scenario

The robust estimator developed in this paper are based on the following uncertain DMJLS

$$\begin{aligned} x_{i+1} =& (F_{i,\Theta_i} + \delta F_{i,\Theta_i})x_i + G_{i,\Theta_i} u_i; \quad i \geq 0; \\ y_i^{(k)} =& (H_{i,\Theta_i}^{(k)} + \delta H_{i,\Theta_i}^{(k)})x_i + D_{i,\Theta_i}^{(k)} v_i^{(k)}; \quad k = 1, ..., L. \end{aligned} \tag{2}$$

where x_i is the $\Re^n$ - valued state, u_i is the $\Re^p$ - random state disturbance, y_i is the $\Re^m$ - valued output sequence, and v_i is the $\Re^q$ - random output disturbance; $F_{i,r} \in \Re^{n\times n}, G_{i,r} \in \Re^{n\times q_1}, H_{i,r}^{(k)} \in \Re^{m\times n}$, and $D_{i,r}^{(k)} \in \Re^{m\times q_2}$ $(r \in \{1, ..., N\})$ are known time varying parameters and dependent of unknown discrete-time Markov chain $\{\Theta_i = r\}$ with transition probability matrix $P = [p_{jr}]$ and probability distribution $\pi_{i,j} := P(\Theta_i = j)$; and $\delta F_{i,r}^{(k)}$ and $\delta H_{i,r}^{(k)}$ are parametric uncertainties. The random disturbances $\{u_i\}$ and $\{v_i\}$ are assumed null mean second-order moments, independent wide sense stationary sequences mutually independent with covariance matrices equal to U_i and $V_i^{(k)}$, with $k = 1, ..., L$, respectively. $x_0 1_{\{\Theta_0 = r\}}$ are random vectors with $\mathbb{E}\left\{x_0 1_{\{\Theta_0=r\}}\right\} = \mu_r$ (where $1_{\{\cdot\}}$ denotes Dirac measure) and $\mathbb{E}\left\{x_0 x_0^T 1_{\{\Theta_0=r\}}\right\} = V_r$; x_0, $\{\Theta_i\}$, $\{u_i\}$ and $\{v_i\}$ are independent.

The uncertainties of (2) are modelled based on the following matrix functions

$$\begin{aligned} \delta F_{i,\Theta_i} &= M^f_{i,\Theta_i}\Delta^f_{i,\Theta_i}N^f_{i,\Theta_i}, \quad \left\|\Delta^f_{i,\Theta_i}\right\| \leq 1, \\ \delta H^{(k)}_{i,\Theta_i} &= M^{h(k)}_{i,\Theta_i}\Delta^h_{i,\Theta_i}N^{h(k)}_{i,\Theta_i} \quad \left\|\Delta^h_{i,\Theta_i}\right\| \leq 1, \end{aligned} \tag{3}$$

where Δ^f_{i,Θ_i} and Δ^h_{i,Θ_i} are unknown matrices which parameterize uncertainties domains for state and measurement matrices and M^f_{i,Θ_i}, N^f_{i,Θ_i}, $M^{h(k)}_{i,\Theta_i}$, $N^{h(k)}_{i,\Theta_i}$ are known matrices of appropriate dimensions. Due to nature of Markov chain, which is considered unknown a-priori, the estimates of System (2) are performed taking into account an augmented state variable which encompasses all possible Markovian modes

$$\begin{aligned} z_i &:= \begin{bmatrix} z^T_{i,1} & \dots & z^T_{i,N} \end{bmatrix}^T \in \Re^{Nn}, \\ z_{i,k} &:= x_i 1_{\{\Theta(i)=k\}} \in \Re^n. \end{aligned} \tag{4}$$

Following [2] and [12], we redefine System (2) based on (4)

$$\begin{aligned} z_{i+1} &:= (\mathcal{F}_i + \delta\mathcal{F}_i)z_i + \psi_i; \quad i \geq 0 \\ y^{(k)}_i &:= (\mathcal{H}^{(k)}_i + \delta\mathcal{H}^{(k)}_i)z_i + \varphi^{(k)}_i, \end{aligned} \tag{5}$$

where the parameter matrices are given by

$$\mathcal{F}_i := \begin{bmatrix} p_{11}F_{i,1} & \dots & p_{N1}F_{i,N} \\ \vdots & \ddots & \vdots \\ p_{1N}F_{i,1} & \dots & p_{NN}F_{i,N} \end{bmatrix} \in \Re^{NnxNn}, \tag{6}$$

$$\begin{aligned} \delta\mathcal{F}_i &:= \begin{bmatrix} p_{11}\delta F_{i,1} & \dots & p_{N1}\delta F_{i,N} \\ \vdots & \ddots & \vdots \\ p_{1N}\delta F_{i,1} & \dots & p_{NN}\delta F_{i,N} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} p_{11}M^f_{i,1} & \dots & p_{N1}M^f_{i,N} \\ \vdots & \ddots & \vdots \\ p_{1N}M^f_{i,1} & \dots & p_{NN}M^f_{i,N} \end{bmatrix}}_{M^f_i} \underbrace{\begin{bmatrix} \Delta^f_{i,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Delta^f_{i,N} \end{bmatrix}}_{\Delta^f_i} \\ &\qquad \underbrace{\begin{bmatrix} N^f_{i,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & N^f_{i,N} \end{bmatrix}}_{N^f_i}, \end{aligned} \tag{7}$$

$$
\begin{aligned}
\mathcal{H}_i^{(k)} &:= \begin{bmatrix} H_{i,1}^{(k)} & \dots & H_{i,N}^{(k)} \end{bmatrix} \in \Re^{m \times Nn}, \\
\delta\mathcal{H}_i^{(k)} &:= \begin{bmatrix} \delta H_{i,1}^{(k)} & \dots & \delta H_{i,N}^{(k)} \end{bmatrix} = \underbrace{\begin{bmatrix} M_{i,1}^{h(k)} & \dots & M_{i,N}^{h(k)} \end{bmatrix}}_{M_i^{h(k)}} \\
&\times \underbrace{\begin{bmatrix} \Delta_{i,1}^{h} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Delta_{i,N}^{h} \end{bmatrix}}_{\Delta_i^h} \underbrace{\begin{bmatrix} N_{i,1}^{h(k)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & N_{i,N}^{h(k)} \end{bmatrix}}_{N_i^{h(k)}}.
\end{aligned}
$$

The random state and output disturbance variances, ψ_i and $\varphi_i^{(k)}$, are given by

$$
R_i^{(k)} := \mathbb{E}\left\{\varphi_i^{(k)}\varphi_i^{(k)T}\right\} = \sum_{j=1}^{N} \pi_{i,j} D_{i,j}^{(k)} V_i^{(k)} D_{i,j}^{(k)T}, \tag{8}
$$

$$
\begin{aligned}
\Pi_i &:= \mathbb{E}\left\{\psi_i\psi_i^T\right\} = \mathrm{diag}\{Z_r^U\} - Z_i^{L'} \\
&+ \mathrm{diag}\left[\sum_{j=1}^{N} p_{jr}\pi_{i,j} G_{i,j} U_i G_{i,j}^T\right],
\end{aligned} \tag{9}
$$

where the superscript U and L stand for upper and lower bounds. The first term Z_r^U can be calculated through the minimization problem (10)

$$
\begin{aligned}
&\min tr\left(Z_r^U\right) \\
&\text{s.t.} \begin{bmatrix} Z_r^U - \sum_{j=1}^{N} p_{jr} F_j Z_j^U F_j^T - \sum_{j=1}^{N} p_{jr}\varepsilon_j M_j^f M_j^{fT} - \mathcal{U}_r & p_{1r}^{1/2} F_1 Z_1^U N_1^{fT} & \dots & p_{Nr}^{1/2} F_N Z_N^U N_N^{fT} \\ p_{1r}^{1/2} N_1^f Z_1^U F_1^T & \varepsilon_1 I - N_1^f Z_1^U N_1^{fT} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{Nr}^{1/2} N_N^f Z_N^U F_N^T & 0 & \dots & \varepsilon_N I - N_N^f Z_N^U N_N^{fT} \end{bmatrix} > 0,
\end{aligned} \tag{10}
$$

where $\mathcal{U}_r := \sum_{j=1}^{N} p_{jr}\pi_j G_j U G_j^T$. The second term $Z_i^{L'}$ can be computed recursively with

$$
Z_{i+1}^{L'} := \mathcal{F}_i\left(Z_i^{L'^{-1}} + \gamma_{2_i}^{-1} N_i^{f^T} N_i^f\right)^{-1} \mathcal{F}_i^T - \gamma_{2_i} M_i^f M_i^{f^T} + \mathrm{diag}\left[\sum_{j=1}^{N} p_{jr}\pi_{i,j} G_{i,j} U_i G_{i,j}^T\right], \tag{11}
$$

where the scalar parameter γ_{2_i} is given by

$$
\gamma_{2_i} := \tau_i \sigma_{min}\left(\mathrm{diag}\left[\sum_{j=1}^{N} p_{jr}\pi_{i,j} G_{i,j} U_i G_{i,j}^T\right]\right)\sigma_{max}\left(M_i^f M_i^{f^T}\right)^{-1}, \quad \text{with} \quad 0 < \tau_i < 1. \tag{12}
$$

The robust estimator was deduced based on solution of the following optimization problem are related with system (5)

$$\min_{\{z_i,z_{i+1}\}} \max_{\{\delta\mathcal{F}_i,\delta\mathcal{H}_{i+1,k}\}} \Bigg[||z_i - \hat{z}_{i|i}||^2_{\tilde{Z}^{-1}_{i|i}} + ||z_{i+1} - (\mathcal{F}_i + \delta\mathcal{F}_i)z_i||^2_{\Pi_i^{-1}}$$
$$+ \sum_{k=1}^{L} ||y^{(k)}_{i+1} - (\mathcal{H}^{(k)}_{i+1} + \delta\mathcal{H}^{(k)}_{i+1})z_{i+1}||^2_{R^{(k)-1}_{i+1}} \Bigg]. \quad (13)$$

2.2 Formulation of the Problem Estimation for DMJLS in Probabilistic Data Fusion

We present the formulation of the problem estimation for DMJLS in probabilistic data fusion scenario. Consider the uncertain measurement equations for $k = 1, ..., L$

$$y_i = (H^{(l)}_{i,\Theta_i} + \delta H^{(l)}_{i,\Theta_i})x_i + D_{i,\Theta_i}v_i; \quad i \geq 0; \quad l = 1, ..., L; \quad (14)$$

where v_i are measurement noises and $\delta H^{(l)}_{i,\Theta_i}$ are uncertainties associated to parameter matrices $H^{(l)}_{i,\Theta_i}$. In this situation, a single measurement y_i is one available that could have risen from a selection of L models with a probability $\overline{\pi}_{i,s}$ for each possible model, as depicted in Fig. (2).

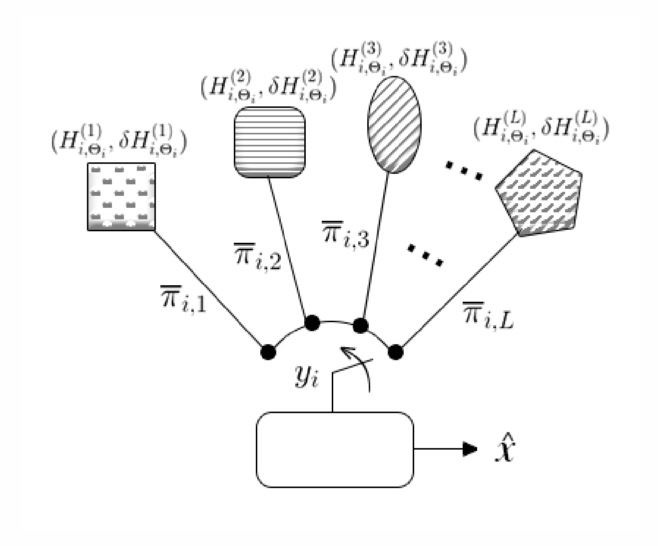

Figure 2: Data Fusion Scenario

Consider the following DMJLS

$$\begin{aligned} x_{i+1} &= (F_{i,\Theta_i} + \delta F_{i,\Theta_i})x_i + G_{i,\Theta_i}u_i; \quad i \geq 0, \\ y_i &= (H^{(l)}_{i,\Theta_i} + \delta H^{(l)}_{i,\Theta_i})x_i + D_{i,\Theta_i}v_i; \quad l = 1, ..., L. \end{aligned} \quad (15)$$

where $H^{(l)}_{i,r} \in \Re^{m\times n}$ ($r \in \{1, ..., N\}$ and $l \in \{1, ..., L\}$) is known parameter-varying in time and dependent of the unknown discrete-time Markov chain $\{\Theta_i\}$; y_i are measurements using one of the matrices $H^{(l)}_{i,r}$ chosen by a time-variant Markov chain $\left\{\overline{\Theta}_i = l\right\}$, in each instant i, according to the transition probability matrix $\overline{P} = [\overline{p}_{sl}]$ with probability distribution $\overline{\pi}_{i,s} := P(\overline{\Theta}_i = s)$

$(s \in \{1, ..., L\})$;. The uncertainty $\delta H^{(l)}_{i,\Theta_i}$ is based on the following matrix functions

$$\delta H^{(l)}_{i,\Theta_i} = M^{h(l)}_{i,\Theta_i} \Delta^h_{i,\Theta_i} N^{h(l)}_{i,\Theta_i} \quad \left\| \Delta^h_{i,\Theta_i} \right\| \leq 1, \tag{16}$$

where Δ^h_{i,Θ_i} is unknown matrix which parameterize the uncertainty domain for the measurement matrix and $M^{h(l)}_{i,\Theta_i}$, $N^{h(l)}_{i,\Theta_i}$ are known matrices of appropriate dimensions.

Deduced as in previous section, the augmented system based on system (15) is given by

$$\begin{aligned} z_{i+1} &:= (\mathcal{F}_i + \delta\mathcal{F}_i) z_i + \psi_i; \qquad i \geq 0, \\ y_i &:= (\mathcal{H}^{(l)}_i + \delta\mathcal{H}^{(l)}_i) z_i + \varphi_i; \quad l = 1, ..., L. \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mathcal{H}^{(l)}_i &:= \begin{bmatrix} H^{(l)}_{i,1} & \cdots & H^{(l)}_{i,N} \end{bmatrix}, \\ \delta\mathcal{H}^{(l)}_i &:= \begin{bmatrix} \delta H^{(l)}_{i,1} & \cdots & \delta H^{(l)}_{i,N} \end{bmatrix} \end{aligned} \tag{18}$$

$$\begin{aligned} &= \underbrace{\begin{bmatrix} M^{h(l)}_{i,1} & \cdots & M^{h(l)}_{i,N} \end{bmatrix}}_{M^{h(l)}_i} \\ &\times \underbrace{\begin{bmatrix} \Delta^h_{i,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta^h_{i,N} \end{bmatrix}}_{\Delta^h_i} \underbrace{\begin{bmatrix} N^{h(l)}_{i,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N^{h(l)}_{i,N} \end{bmatrix}}_{N^{h(l)}_i}, \end{aligned} \tag{19}$$

$$R_i := \mathbb{E}\left\{ \varphi_i \varphi_i^T \right\} = \sum_{j=1}^{N} \pi_{i,j} D_{i,j} V_i D^T_{i,j}. \tag{20}$$

The robust estimator was deduced based on solution of the following optimization problem related with System (17):

$$\begin{aligned} \min_{\{z_i, z_{i+1}\}} \max_{\{\delta\mathcal{F}_i, \delta\mathcal{H}^{(l)}_{i+1}\}} & \Bigg[||z_i - \hat{z}_{i|i}||^2_{\tilde{Z}^{-1}_{i|i}} + ||z_{i+1} - (\mathcal{F}_i + \delta\mathcal{F}_i) z_i||^2_{\Pi^{-1}_i} \\ &+ \sum_{l=1}^{L} ||y_{i+1} - (\mathcal{H}^{(l)}_{i+1} + \delta\mathcal{H}^{(l)}_{i+1}) z_{i+1}||^2_{R^{-1}_{i+1} \overline{\pi}^{-1}_{i+1,l}} \Bigg]. \end{aligned} \tag{21}$$

The solutions of the optimization problems (13) and (21) will be apresented in the next section.

3 ROBUST ESTIMATION FOR DMJLS IN DATA FUSION SCENARIOS

Based on the augmented DMJLS in data fusion scenarios presented in previous section we present the robust estimation approaches for the DMJLS in weighted and probabilistic data fusions. They are deduced through minimization of the worst-possible regularized residual norm.

To solve the problem (13) and (21) it is necessary enunciate the following Lemma, proposed by [19].

Lemma 3.1. *[19] Consider the problem*

$$\hat{x} = arg \min_{x} \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \left[||x||_Q^2 + \sum_{k=1}^{L} ||(A_k + \delta A_k)x - (b_k + \delta b_k)||_{W_k}^2 \right]. \tag{22}$$

where $||\alpha||_\Sigma^2 = \alpha^T \Sigma \alpha$ is the weighted norm. Assume the uncertainties satisfy the constraints

$$\begin{bmatrix} \delta A_k & \delta b_k \end{bmatrix} = G_k \Delta \begin{bmatrix} N_{a,k} & N_{b,k} \end{bmatrix}. \tag{23}$$

where A_k are the data matrices, b_k are the measurement vectors which are assumed to be known, x is the unknown vector and $Q = Q^T > 0$ and $W_k = W_k^T \geq 0$ are given weighting matrices; $G_k, N_{a,k}$ and $N_{b,k}$ are matrices of appropriate dimensions known in the problem and Δ an arbitrary contraction matrix, such that $||\Delta|| \leq 1$.

The solution of the problem (22) can be determined as follows

$$\hat{x} = \left[\widehat{Q} + \sum_{k=1}^{L} A_k^T \widehat{W}_k A_k \right]^{-1} \sum_{k=1}^{L} \left[A_k^T \widehat{W}_k b_k + \hat{\lambda}_k N_{a,k}^T N_{b,k} \right]. \tag{24}$$

The modified weighting matrices $\{\widehat{Q}, \widehat{W}_k\}$ computed from given weighting matrices $\{Q, W_k\}$ in terms of the scalars $\{\hat{\lambda}_k\}$, as follows

$$\widehat{Q} := Q + \sum_{k=1}^{L} \hat{\lambda}_k N_{a,k}^T N_{b,k}, \tag{25}$$

$$\widehat{W}_k := W_k + W_k G_k (\hat{\lambda}_k I - G_k^T W_k G_k)^{\dagger} G_k^T W_k. \tag{26}$$

For this, is necessary determines the scalars $\{\hat{\lambda}_1, ..., \hat{\lambda}_L\}$ that minimize the function

$$G(\lambda_k) := \left[||x^o||_Q^2 + \sum_{k=1}^{L} C_k(x^o, \lambda_k) \right], \quad k \in \{1, ..., L\} \tag{27}$$

over the interval $\lambda_k \geq ||G_k^T W_k G_k||$, where λ_k a non-negative number.

Based on Lemma 3.1 the robust estimation's algorithms for DMJLS in data fusion scenarios will be deduced in the next section.

3.1 Robust Estimation for DMJLS in Weighted Data Fusion

Based on the Lemma 3.1 we present the robust estimation approach for the DMJLS in weighted data fusions in the next Theorem.

Theorem 3.2. *The robust estimate of z_i in weighted data fusion scenario, given by $\hat{z}_{i|i}$, which can be obtained by the following recursive algorithm:* ***Step*** 0***:*** *(Initial conditions):*

$$\hat{R}_0^{(k)} := R_0^{(k)} - \hat{\lambda}_{-1}^{(k)^{-1}} M_0^{h(k)} M_0^{h(k)^T}, \tag{28}$$

$$\bar{R}_0 := \operatorname{diag}\left[\hat{R}_0^{(k)}\right], \tag{29}$$

$$\bar{N}_0^h := \begin{bmatrix} \sqrt{\hat{\lambda}_{-1}^{(1)}} N_0^{h(1)} \\ \sqrt{\hat{\lambda}_{-1}^{(2)}} N_0^{h(2)} \\ \vdots \\ \sqrt{\hat{\lambda}_{-1}^{(L)}} N_0^{h(L)} \end{bmatrix}, \tag{30}$$

$$\tilde{Z}_{0|0} := (P_0^{-1} + \bar{\mathcal{H}}_0^T \bar{R}_0^{-1} \bar{\mathcal{H}}_0 + \bar{N}_0^{h^T} \bar{N}_{h,0})^{-1}, \tag{31}$$

$$\hat{z}_{0|0} := \tilde{Z}_{0|0} \bar{\mathcal{H}}_0^T \bar{R}_0^{-1} \bar{z}_0. \tag{32}$$

Step 0***:*** *Compute $R_{i+1}^{(k)}$, Π_i, $Z_{i,j}^U$ and $Z_{i,j}^L$ through (8), (9), (10), and (11), respectively.*

Step 1***:*** *If $M_i^f = 0$ and $M_{i+1}^h = 0$, then $\hat{\lambda}_i^{(k)} = 0$. Otherwise, it is chosen as*

$$\hat{\lambda}_i^{(k)} = (1 + \rho_{1i}) \left\| \begin{bmatrix} \sqrt{L} M_i^{f^T} & 0 \\ 0 & M_{i+1}^{h(k)^T} \end{bmatrix} \begin{bmatrix} \frac{1}{L}\Pi_i^{-1} & 0 \\ 0 & R_{i+1}^{(k)^{-1}} \end{bmatrix} \begin{bmatrix} \sqrt{L} M_i^f & 0 \\ 0 & M_{i+1}^{h(k)} \end{bmatrix} \right\| \tag{33}$$

where $\rho_{1_i} > 0$ and parameters $\{R_{i+1}^{(k)}, \Pi_i\}$ are changed to corrected parameters

$$\begin{aligned} \hat{R}_{i+1}^{(k)} &:= R_{i+1}^{(k)} - \hat{\lambda}_i^{(k)^{-1}} M_{i+1}^{h(k)} M_{i+1}^{h(k)^T}, \\ \hat{\Pi}_i^{(k)} &:= \Pi_i - \hat{\lambda}_i^{(k)^{-1}} M_i^f M_i^{f^T}. \end{aligned} \tag{34}$$

Step 2***:*** *If $\hat{\lambda}_i^{(k)} \neq 0$, substitute parameters $\hat{\Pi}_i^{(k)}$, $\hat{R}_{i+1}^{(k)}$, $\mathcal{F}_i$, $\mathcal{H}_{i+1}^{(k)}$, I, by corrected parameters*

$$\begin{aligned} X_i &:= \begin{bmatrix} \Phi_i & 0 \\ 0 & I \end{bmatrix}, \Phi_i := \sum_{k=1}^{L} \frac{1}{L} \hat{\Pi}_i^{(k)^{-1}}, \\ Y_{i+1} &:= \begin{bmatrix} \bar{R}_{i+1} & 0 \\ 0 & I \end{bmatrix}, \bar{R}_{i+1} := \operatorname{diag}\left[\hat{R}_{i+1}^{(k)}\right], \end{aligned} \tag{35}$$

$$\hat{\mathcal{F}}_i := \begin{bmatrix} \mathcal{F}_i \\ \sqrt{\bar{\lambda}_i} N_i^f \end{bmatrix}, \bar{\lambda}_i := \sum_{k=1}^{L} [\hat{\lambda}_i^{(k)}],$$

$$\hat{\mathcal{H}}_{i+1} := \begin{bmatrix} \bar{\mathcal{H}}_{i+1} \\ \bar{N}_{i+1}^h \end{bmatrix}, \bar{\mathcal{H}}_{i+1} := \begin{bmatrix} \mathcal{H}_{i+1}^{(1)} \\ \mathcal{H}_{i+1}^{(2)} \\ \vdots \\ \mathcal{H}_{i+1}^{(L)} \end{bmatrix}$$

$$\bar{N}_{i+1}^h := \begin{bmatrix} \sqrt{\hat{\lambda}_{i,1}} N_{i+1}^{h(1)} \\ \sqrt{\hat{\lambda}_{i,2}} N_{i+1}^{h(2)} \\ \vdots \\ \sqrt{\hat{\lambda}_{i,L}} N_{i+1}^{h(L)} \end{bmatrix}, \hat{I} := \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

***Step** 3: Actualize $\{\tilde{Z}_{i|i}, \hat{z}_{i|i}\}$ with $\{\tilde{Z}_{i+1|i+1}, \hat{z}_{i+1|i+1}\}$ through the following recursive equations*

$$\tilde{Z}_{i+1|i+1} := \left(\begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (X_i^{-1} + \hat{\mathcal{F}}_i \tilde{Z}_{i|i} \hat{\mathcal{F}}_i^T) & 0 \\ 0 & Y_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix} \right)^{-1}, \tag{36}$$

$$\hat{z}_{i+1|i+1} := \tilde{Z}_{i+1|i+1} \begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (X_i^{-1} + \hat{\mathcal{F}}_i \tilde{Z}_{i|i} \hat{\mathcal{F}}_i^T) & 0 \\ 0 & Y_{i+1} \end{bmatrix}^{-1} \times \begin{bmatrix} \hat{\mathcal{F}}_i \hat{z}_{i|i} \\ \Sigma_{i+1} \end{bmatrix}, \tag{37}$$

where

$$\Sigma_{i+1} := \begin{bmatrix} \bar{y}_{i+1} \\ 0 \end{bmatrix}, \bar{y}_{i+1} := \begin{bmatrix} y_{i+1}^{(1)} \\ y_{i+1}^{(2)} \\ \vdots \\ y_{i+1}^{(L)} \end{bmatrix}. \tag{38}$$

***Step** 4: From $\hat{z}_{i+1|i} := \left[\hat{z}_{i+1,1|i}^T \dots \hat{z}_{i+1,N|i}^T\right]^T$, compute $\hat{x}_{i+1|i} = \sum_{j=1}^{N} \hat{z}_{i+1,j|i}$.*

Proof: The optimization problem (13) can be rewritten as

$$\hat{x} = arg \min_{x} \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \left[||x||_Q^2 + \sum_{k=1}^{L} ||(A_k + \delta A_k)x - (b_k + \delta b_k)||_{W_k}^2 \right], \tag{39}$$

$$\begin{bmatrix} \delta A_k & \delta b_k \end{bmatrix} = G_k \Delta \begin{bmatrix} N_{a,k} & N_{b,k} \end{bmatrix}, \tag{40}$$

considering the following identifications

$$
\begin{gathered}
x \leftarrow \begin{bmatrix} z_i - \hat{z}_{i|i} \\ z_{i+1} \end{bmatrix}, Q \leftarrow \begin{bmatrix} \tilde{Z}_{i|i}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
W_k \leftarrow \begin{bmatrix} \frac{1}{L}\Pi_i^{-1} & 0 \\ 0 & R_{i+1}^{(k)^{-1}} \end{bmatrix}, b_k \leftarrow \begin{bmatrix} \mathcal{F}_i \hat{z}_{i|i} \\ z_{i+1} \end{bmatrix}, \\
\delta b_k \leftarrow \begin{bmatrix} \delta \mathcal{F}_i \hat{z}_{i|i} \\ 0 \end{bmatrix}, \delta A_k \leftarrow \begin{bmatrix} -\delta \mathcal{F}_i & I \\ 0 & \delta \mathcal{H}_{i+1}^{(k)} \end{bmatrix}, \\
A_k \leftarrow \begin{bmatrix} -\mathcal{F}_i & I \\ 0 & \mathcal{H}_{i+1}^{(k)} \end{bmatrix}, G_k \leftarrow \begin{bmatrix} \sqrt{L} M_i^f & 0 \\ 0 & M_{i+1}^{h(k)} \end{bmatrix}, \\
N_{a,k} \leftarrow \begin{bmatrix} -N_i^f & 0 \\ 0 & N_{i+1}^{h(k)} \end{bmatrix}, N_{b,k} \leftarrow \begin{bmatrix} N_i^f \hat{z}_{i|i} \\ 0 \end{bmatrix}, \\
\Delta \leftarrow \begin{bmatrix} \frac{1}{\sqrt{L}} \Delta_i^f & 0 \\ 0 & \Delta_{i+1}^h \end{bmatrix}.
\end{gathered}
$$

According to [19], the solution of the optimization problem (65) is given by

$$
\hat{x} = \left[\widehat{Q} + \sum_{k=1}^{L} A_k^T \widehat{W}_k A_k \right]^{-1} \sum_{k=1}^{L} \left[A_k^T \widehat{W}_k b_k + \hat{\lambda}_k N_{a,k}^T N_{b,k} \right], \tag{41}
$$

where modified weighting matrices $\{\widehat{Q}, \widehat{W}_k\}$, computed from given weighting matrices $\{Q, W_k\}$, are defined by

$$
\widehat{Q} := Q + \sum_{k=1}^{L} \hat{\lambda}_k N_{a,k}^T N_{b,k}, \tag{42}
$$

$$
\widehat{W}_k := W_k + W_k G_k (\hat{\lambda}_k I - G_k^T W_k G_k)^{\dagger} G_k^T W_k. \tag{43}
$$

For this, it is necessary determine the non negative scalars parameters $\{\hat{\lambda}_1, ..., \hat{\lambda}_L\}$ which satisfy $\hat{\lambda}_k \geq ||G_k^T W_k G_k||, k = 1, 2, ..., L$. After some algebras, the solution can been reached.

3.2 Robust Estimation for DMJLS in Probabilistic Data Fusion

Now, we are going to deduce the robust estimator for DMJLS in probabilistic data fusion scenario. Consider the optimization problem

$$
\hat{x} = arg \min_x \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \mathrm{E} \left[||x||_Q^2 + ||(A_{\overline{\Theta}_i} + \delta A_{\overline{\Theta}_i}) x - (b + \delta b_{\overline{\Theta}_i})||_W^2 \right], \tag{44}
$$

where E is the expected value and the parameters $A_{\overline{\Theta}_i}$, $\delta A_{\overline{\Theta}_i}$ and $\delta b_{\overline{\Theta}_i}$ are chosen by a time-variant Markov chain $\left\{\overline{\Theta}_i = l\right\}$ in each instant i, according to transition probability matrix

$\overline{P} = [\overline{p}_{sl}]$ with probability distribution $\overline{\pi}_{i,s} := P(\overline{\Theta}_i = s)$ $(s \in \{1, ..., L\})$.

Considering Ω the sample space of the events the following property holds $1 = \mathbf{1}_\Omega$, being $\mathbf{1}_{(\cdot)}$ the dirac measure. Observe that, (44) can be rewritten as

$$\hat{x} := arg \min_x \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \Bigg[||x||_Q^2 + \mathrm{E}\{||(A_{\overline{\Theta}_i} + \delta A_{\overline{\Theta}_i})x - (b + \delta b_{\overline{\Theta}_i})||_W^2 \times 1_{\{\Theta_i = 1\} \cup ... \cup \{\Theta_i = L\}}\} \Bigg]. \tag{45}$$

Since $\{\overline{\Theta}_i = 1\}, ..., \{\overline{\Theta}_i = L\}$ are disjointed

$$\hat{x} := arg \min_x \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \Bigg[||x||_Q^2 + \mathrm{E}\{||(A_{\overline{\Theta}_i} + \delta A_{\overline{\Theta}_i})x - (b + \delta b_{\overline{\Theta}_i})||_W^2 \times \left(1_{\{\Theta_i = 1\}} + ... + 1_{\{\Theta_i = L\}}\right)\} \Bigg] \tag{46}$$

After some algebra and applying the expected value in (46), it can be rewritten as

$$\hat{x} := arg \min_x \max_{\substack{\{\delta A_l\} \\ \{\delta b_l\}}} \left[||x||_Q^2 + \sum_{l=1}^{L} ||(A_l + \delta A_l)x - (b + \delta b_l)||_{\overline{W}_l}^2 \right] \tag{47}$$

where

$$\begin{aligned} \overline{W}_l &:= W\overline{\pi}_{i,l} \\ \begin{bmatrix} \delta A_l & \delta b_l \end{bmatrix} &:= G_l \Delta \begin{bmatrix} N_{a,l} & N_{b,l} \end{bmatrix}. \end{aligned} \tag{48}$$

The optimization problem (21) can be rewritten as (47). Based on the optimization problem (47) and the Lemma 3.1 we present the robust estimation approach for the DMJLS in probabilistic data fusions in the next Theorem.

Theorem 3.3. *The robust estimate of z_i in probabilistic data fusion scenario, given by $\hat{z}_{i|i}$, which can be obtained by the following recursive algorithm:*

Step* 0: *(Initial conditions):

$$\hat{R}_0^{(l)} := R_0\overline{\pi}_{0,l} - \hat{\lambda}_{-1}^{(l)^{-1}} M_0^{h(l)} M_0^{h(l)^T}, \tag{49}$$

$$\bar{R}_0 := \mathrm{diag}\left[\hat{R}_0^{(l)}\right], \tag{50}$$

$$\bar{N}_0^h := \begin{bmatrix} \sqrt{\hat{\lambda}_{-1}^{(1)}} N_0^{h(1)} \\ \sqrt{\hat{\lambda}_{-1}^{(2)}} N_0^{h(2)} \\ \vdots \\ \sqrt{\hat{\lambda}_{-1}^{(L)}} N_0^{h(L)} \end{bmatrix}, \tag{51}$$

$$\tilde{Z}_{0|0} := (P_0^{-1} + \bar{\mathcal{H}}_0^T \bar{R}_0^{-1} \bar{\mathcal{H}}_0 + \bar{N}_0^{h^T} \bar{N}_{h,0})^{-1}, \tag{52}$$

$$\hat{z}_{0|0} := \tilde{Z}_{0|0} \bar{\mathcal{H}}_0^T \bar{R}_0^{-1} \bar{z}_0. \tag{53}$$

***Step* 1:** *Compute R_{i+1}, Π_i, $Z_{i,j}^U$ and $Z_{i,j}^L$ through (20), (9), (10), and (11), respectively.*

***Step* 2:** *If $M_i^f = 0$ and $M_{i+1}^h = 0$, then $\hat{\lambda}_i^{(k)} = 0$. Otherwise, it is chosen as*

$$\hat{\lambda}_i^{(l)} = (1 + \rho_{1i}) \left\| \begin{bmatrix} \sqrt{L} M_i^{f^T} & 0 \\ 0 & M_{i+1}^{h(l)^T} \end{bmatrix} \begin{bmatrix} \frac{1}{L}\Pi_i^{-1} & 0 \\ 0 & R_{i+1}^{-1} \overline{\pi}_{i+1,l}^{-1} \end{bmatrix} \begin{bmatrix} \sqrt{L} M_i^f & 0 \\ 0 & M_{i+1}^{h(l)} \end{bmatrix} \right\| \tag{54}$$

where $\rho_{1i} > 0$ and the parameters $\{R_{i+1}^{(l)}, \Pi_i\}$ are changed to corrected parameters

$$\hat{R}_{i+1}^{(l)} := R_{i+1} \overline{\pi}_{i+1,l} - \hat{\lambda}_i^{(l)^{-1}} M_{i+1}^{h(l)} M_{i+1}^{h(l)^T}, \tag{55}$$

$$\hat{\Pi}_i^{(l)} := \Pi_i - \hat{\lambda}_i^{(l)^{-1}} M_i^f M_i^{f^T}. \tag{56}$$

***Step* 2:** *If $\hat{\lambda}_i^{(l)} \neq 0$, substitute parameters $\hat{\Pi}_i^{(l)}$, $\hat{R}_{i+1}^{(l)}$, $\mathcal{F}_i$, $\mathcal{H}_{i+1}^{(l)}$, I, by corrected parameters:*

$$X_i := \begin{bmatrix} \Phi_i & 0 \\ 0 & I \end{bmatrix}, \Phi_i := \sum_{k=1}^{L} \frac{1}{L} \hat{\Pi}_i^{(l)^{-1}}, \tag{57}$$

$$Y_{i+1} := \begin{bmatrix} \bar{R}_{i+1} & 0 \\ 0 & I \end{bmatrix}, \bar{R}_{i+1} := \operatorname{diag}\left[\hat{R}_{i+1}^{(l)}\right], \tag{58}$$

$$\hat{\mathcal{F}}_i := \begin{bmatrix} \mathcal{F}_i \\ \sqrt{\bar{\lambda}_i} N_i^f \end{bmatrix}, \bar{\lambda}_i := \sum_{k=1}^{L} [\hat{\lambda}_i^{(l)}], \tag{59}$$

$$\hat{\mathcal{H}}_{i+1} := \begin{bmatrix} \bar{\mathcal{H}}_{i+1} \\ \bar{N}_{i+1}^h \end{bmatrix}, \bar{\mathcal{H}}_{i+1} := \begin{bmatrix} \mathcal{H}_{i+1}^{(1)} \\ \mathcal{H}_{i+1}^{(2)} \\ \vdots \\ \mathcal{H}_{i+1}^{(L)} \end{bmatrix} \tag{60}$$

$$\bar{N}_{i+1}^{h} := \begin{bmatrix} \sqrt{\hat{\lambda}_{i,1}} N_{i+1}^{h(1)} \\ \sqrt{\hat{\lambda}_{i,2}} N_{i+1}^{h(2)} \\ \vdots \\ \sqrt{\hat{\lambda}_{i,L}} N_{i+1}^{h(L)} \end{bmatrix}, \hat{I} := \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{61}$$

***Step** 3: Actualize $\{\tilde{Z}_{i|i}, \hat{z}_{i|i}\}$ with $\{\tilde{Z}_{i+1|i+1}, \hat{z}_{i+1|i+1}\}$ through the following recursive equations*

$$\tilde{Z}_{i+1|i+1} := \left(\begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (X_i^{-1} + \hat{\mathcal{F}}_i \tilde{Z}_{i|i} \hat{\mathcal{F}}_i^T) & 0 \\ 0 & Y_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix} \right)^{-1}, \tag{62}$$

$$\hat{z}_{i+1|i+1} := \tilde{Z}_{i+1|i+1} \begin{bmatrix} \hat{I} \\ \hat{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (X_i^{-1} + \hat{\mathcal{F}}_i \tilde{Z}_{i|i} \hat{\mathcal{F}}_i^T) & 0 \\ 0 & Y_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathcal{F}}_i \hat{z}_{i|i} \\ \Sigma_{i+1} \end{bmatrix}. \tag{63}$$

where

$$\Sigma_{i+1} := \begin{bmatrix} \bar{y}_{i+1} \\ 0 \end{bmatrix}, \bar{y}_{i+1} := \begin{bmatrix} y_{i+1} \\ y_{i+1} \\ \vdots \\ y_{i+1} \end{bmatrix}. \tag{64}$$

***Step** 4: From $\hat{z}_{i+1|i} := \left[\hat{z}_{i+1,1|i}^T \dots \hat{z}_{i+1,N|i}^T \right]^T$, compute $\hat{x}_{i+1|i} = \sum_{j=1}^{N} \hat{z}_{i+1,j|i}$.*

Proof: The optimization problem (21) can be rewritten as

$$\hat{x} = arg \min_{x} \max_{\substack{\{\delta A_k\} \\ \{\delta b_k\}}} \left[||x||_Q^2 + \sum_{k=1}^{L} ||(A_k + \delta A_k)x - (b_k + \delta b_k)||_{W_k}^2 \right], \tag{65}$$

$$\begin{bmatrix} \delta A_k & \delta b_k \end{bmatrix} = G_k \Delta \begin{bmatrix} N_{a,k} & N_{b,k} \end{bmatrix}, \tag{66}$$

considering the following identifications

$$
\begin{aligned}
x \leftarrow \begin{bmatrix} z_i - \hat{z}_{i|i} \\ z_{i+1} \end{bmatrix}, Q \leftarrow \begin{bmatrix} \tilde{Z}_{i|i}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\overline{W}_l \leftarrow \begin{bmatrix} \frac{1}{L}\Pi_i^{-1} & 0 \\ 0 & R_{i+1}^{-1}\overline{\pi}_{i+1,l}^{-1} \end{bmatrix}, b \leftarrow \begin{bmatrix} \mathcal{F}_i \hat{z}_{i|i} \\ z_{i+1} \end{bmatrix}, \\
\delta b_l \leftarrow \begin{bmatrix} \delta\mathcal{F}_i \hat{z}_{i|i} \\ 0 \end{bmatrix}, \delta A_l \leftarrow \begin{bmatrix} -\delta\mathcal{F}_i & I \\ 0 & \delta\mathcal{H}_{i+1}^{(l)} \end{bmatrix}, \\
A_l \leftarrow \begin{bmatrix} -\mathcal{F}_i & I \\ 0 & \mathcal{H}_{i+1}^{(l)} \end{bmatrix}, G_l \leftarrow \begin{bmatrix} \sqrt{L}M_i^f & 0 \\ 0 & M_{i+1}^{h(l)} \end{bmatrix}, \\
N_{a,l} \leftarrow \begin{bmatrix} -N_i^f & 0 \\ 0 & N_{i+1}^{h(l)} \end{bmatrix}, N_{b,l} \leftarrow \begin{bmatrix} N_i^f \hat{z}_{i|i} \\ 0 \end{bmatrix}, \\
\Delta \leftarrow \begin{bmatrix} \frac{1}{\sqrt{L}}\Delta_i^f & 0 \\ 0 & \Delta_{i+1}^h \end{bmatrix}.
\end{aligned}
\tag{67}
$$

According to [19], the solution of optimization problem (47) is given by

$$
\hat{x} = \left[\widehat{Q} + \sum_{l=1}^{L} A_l^T \widehat{W}_l A_l\right]^{-1} \sum_{l=1}^{L} \left[A_l^T \widehat{W}_l b + \hat{\lambda}_l N_{a,l}^T N_{b,l}\right]. \tag{68}
$$

where modified weighting matrices $\{\widehat{Q}, \widehat{W}_l\}$, computed from given weighting matrices $\{Q, W_l\}$, are defined by

$$
\widehat{Q} := Q + \sum_{l=1}^{L} \hat{\lambda}_l N_{a,l}^T N_{b,l}, \tag{69}
$$

$$
\widehat{W}_l := W_l + W_l G_l (\hat{\lambda}_l I - G_l^T W_l G_l)^{\dagger} G_l^T W_l. \tag{70}
$$

For this, it is necessary determine the non negative scalars parameters $\{\hat{\lambda}_1, ..., \hat{\lambda}_L\}$ which satisfy $\hat{\lambda}_l \geq ||G_l^T W_l G_l||, l = 1, 2, ..., L$. After some algebras, the solution can been reached.

Remark 3.4. It is noteworthy that when systems (2) and (15) have no uncertainties, i.e., when $M_{f,i} = 0$, $M_{h,i+1}^{(k)} = 0$ $(M_{h,i+1}^{(l)} = 0)$, $N_{e,i+1} = 0$, $N_{f,i} = 0$ and $N_{h,i+1}^{(k)} = 0$ $(N_{h,i+1}^{(l)} = 0)$, from (36)-(37) and (62)-(63),it is obtained the following nominal filters. For weighted data fusion,

$$
\tilde{Z}_{i+1|i+1} := \left(\begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (\Pi_i + \mathcal{F}_i \tilde{Z}_{i|i} \mathcal{F}_i^T) & 0 \\ 0 & \bar{R}_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix} \right)^{-1}, \tag{71}
$$

$$
\hat{z}_{i+1|i+1} := \tilde{Z}_{i+1|i+1} \begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (\Pi_i + \mathcal{F}_i \tilde{Z}_{i|i} \mathcal{F}_i^T) & 0 \\ 0 & \bar{R}_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{F}_i \hat{z}_{i|i} \\ \bar{y}_{i+1} \end{bmatrix}, \tag{72}
$$

where

$$\bar{\mathcal{H}}_{i+1} := \begin{bmatrix} \mathcal{H}_{i+1}^{(1)} \\ \mathcal{H}_{i+1}^{(2)} \\ \vdots \\ \mathcal{H}_{i+1}^{(L)} \end{bmatrix}, \bar{y}_{i+1} := \begin{bmatrix} y_{i+1}^{(1)} \\ y_{i+1}^{(2)} \\ \vdots \\ y_{i+1}^{(L)} \end{bmatrix}$$
$$\bar{R}_{i+1} := \operatorname{diag}\left[R_{i+1}^{(k)}\right].$$

For probabilistic data fusion,

$$\tilde{Z}_{i+1|i+1} := \left(\begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (\Pi_i + \mathcal{F}_i \tilde{Z}_{i|i} \mathcal{F}_i^T) & 0 \\ 0 & \mathcal{R}_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix} \right)^{-1}, \tag{73}$$
$$\hat{z}_{i+1|i+1} := \tilde{Z}_{i+1|i+1} \begin{bmatrix} I \\ \bar{\mathcal{H}}_{i+1} \end{bmatrix}^T \begin{bmatrix} (\Pi_i + \mathcal{F}_i \tilde{Z}_{i|i} \mathcal{F}_i^T) & 0 \\ 0 & \mathcal{R}_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{F}_i \hat{z}_{i|i} \\ \bar{y}_{i+1} \end{bmatrix}, \tag{74}$$

where

$$\bar{\mathcal{H}}_{i+1} := \begin{bmatrix} \mathcal{H}_{i+1}^{(1)} \\ \mathcal{H}_{i+1}^{(2)} \\ \vdots \\ \mathcal{H}_{i+1}^{(L)} \end{bmatrix}, \bar{y}_{i+1} := \begin{bmatrix} y_{i+1} \\ y_{i+1} \\ \vdots \\ y_{i+1} \end{bmatrix}, \tag{75}$$
$$\mathcal{R}_{i+1} := \operatorname{diag}\left[R_{i+1}\pi_{i+1,l}\right].$$

Remark 3.5. When $k = 1$ and $l = 1$ are considered, i.e., systems (2) and (15) have one unique measurement equation, it is easy to see that proposed filters in this technical note assume similar formulation to the Recursive Robust Filter proposed in [12].

4 NUMERICAL EXAMPLES

In the numerical examples, a comparative study is presented to show the effectiveness of DMJLS robust estimator in data fusion scenarios. We consider the following probability transition matrix and parameters

$$\begin{gathered} P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}, F_1 = \begin{bmatrix} 0.7 & 0 \\ 0.1 & 0.2 \end{bmatrix}, F_2 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \\ G_1 = G_2 = \begin{bmatrix} 0.8731 & 0 \\ 0 & 0.2089 \end{bmatrix}, M_1^f = \begin{bmatrix} 0.13 & 0 \\ 0.13 & 0.13 \end{bmatrix}, \\ M_2^f = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.13 \end{bmatrix}, N_1^f = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \\ N_2^f = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, M_1^{h(1)} = M_2^{h(1)} = \begin{bmatrix} 0.39 & 0 \end{bmatrix}, \end{gathered} \tag{76}$$

$$\begin{gathered}
M_1^{h(2)} = M_2^{h(2)} = \begin{bmatrix} 0.38 & 0 \end{bmatrix}, \\
M_1^{h(3)} = M_2^{h(3)} = \begin{bmatrix} 0.37 & 0 \end{bmatrix}, \\
N_1^{h(1)} = \begin{bmatrix} 3 & 5 \\ 0 & 0.01 \end{bmatrix}, N_2^{h(1)} = \begin{bmatrix} 3 & 0 \\ 0 & 0.01 \end{bmatrix}, \\
N_1^{h(2)} = N_2^{h(2)} = \begin{bmatrix} 1.4 & 0 \\ 0 & 6 \end{bmatrix}, N_1^{h(3)} = N_2^{h(3)} = \begin{bmatrix} 1.3 & 0 \\ 0 & 5 \end{bmatrix}, \\
H_1^{(1)} = H_2^{(1)} = \begin{bmatrix} 0.01 & 0 \end{bmatrix}, H_1^{(2)} = H_2^{(2)} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\
H_1^{(3)} = H_2^{(3)} = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, D_1^{(1)} = D_2^{(1)} = 0.008, \\
D_1^{(2)} = D_2^{(2)} = 0.007, D_1^{(3)} = D_2^{(3)} = 0.006,
\end{gathered} \tag{77}$$

for values of Θ_i randomly generated. The initial condition x_0 is considered Gaussian with mean $\begin{bmatrix} 0.196 & 0.295 \end{bmatrix}^T$ and variance $\begin{bmatrix} 0.0384 & 0.0578 \\ 0.0578 & 0.870 \end{bmatrix}$, $\Theta_i \in \{1, 2\}$, $\bar{\pi}_1(0) = 0.05$ e $\bar{\pi}_2(0) = 0.95$.

First, we present a comparative study between robust estimator proposed in [12], that consider just one measurement model, and the robust estimator in weighted data fusion scenario. The root mean square error (rms) was simulated. Were considered $i = 0, \ldots, 100$ iterations for each recursive step of 4000 Monte Carlo simulations. Both robust estimator were simulated considering that the parameter matrix $H_i^{(1)}$ is simulating a modelling error. Notice in Figure (3) the advantage of using DMJLS robust estimator in weighted data fusion scenario when there is one parameter simulating modelling error.

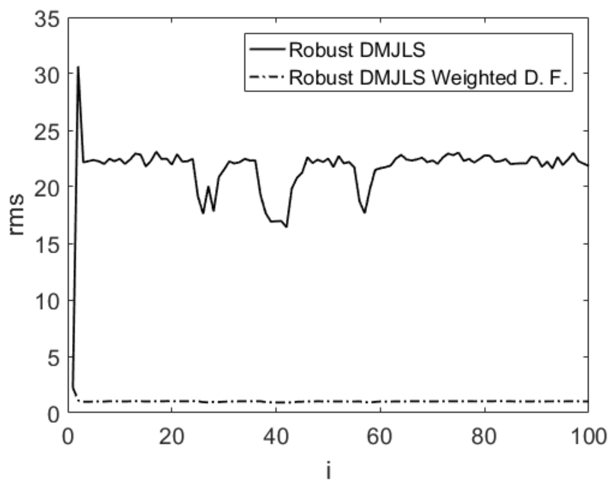

Figure 3: Comparison between the Robust Estimator of [12] and the Robust Estimator in Weighted Data Fusion Scenario.

Second, in the same way, we present a comparative study between robust estimator proposed in [12] and the robust estimator in probabilistic data fusion scenario. Were considered the following initial values for the probability distribution in the robust estimator in probabilistic data fusion

$$\bar{\pi}_1(0) = 0.05, \quad \bar{\pi}_2(0) = 0.475, \quad \bar{\pi}_3(0) = 0.475. \tag{78}$$

The robust estimator in probabilistic data fusion outperforms the robust estimator proposed in [12] as shown in Figure (4).

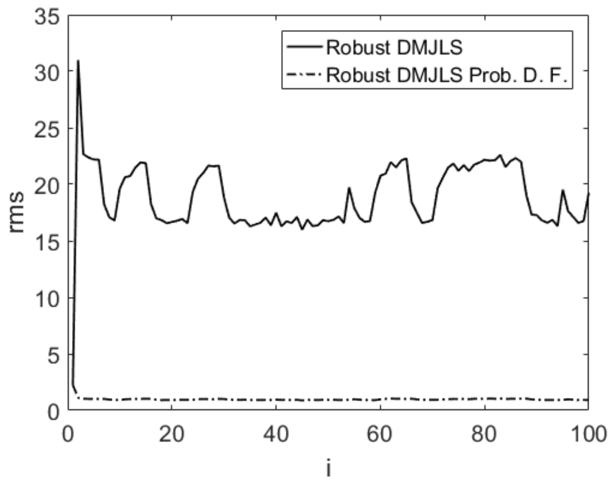

Figure 4: Comparison between DMJLS Estimator of [12] and DMJLS Estimator in Probabilistic Data Fusion Scenario.

5 CONCLUSION

In this paper were developed robust estimates in a data fusion scenario, for the both cases: weighted and probabilistic data fusion. The estimates were deduced based on the technique proposed in [19]. Numerical example showed the new approaches effectiveness when there is at least one parameter simulating modelling error.

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