DOI: 10.22481/intermaths.v3i1.10862
Licença Creative Commons
Article

INTERMATHS
ISSN 2675-8318
Volume 3, Number 1 (2022), pp. 37 – 49

Fractional operators with Kaniadakis logarithm kernels

Ana Paula Perovano

Universidade Estadual do Sudoeste da Bahia, Vitória da Conquista, BA, Brasil

apperovano@uesb.edu.br

Fernando Santos Silva

Universidade Estadual do Sudoeste da Bahia, Vitória da Conquista, BA, Brasil

fssilva@uesb.edu.br

Logaritmo de Kaniadakis como núcleo de operadores fracionários

Resumo

Neste artigo, são propostos novos tipos de operadores fracionários com núcleo logarítmico κ-deformado. Analisamos esses operadores e provamos vários fatos sobre eles, incluindo uma propriedade de semigrupo. Os resultados da existência são estabelecidos em espaços funcionais apropriados. Provamos que esses resultados são válidos de uma só vez para vários operadores fracionários clássicos, como os operadores de Riemann-Liouville, Caputo e os operadores de Hadamard dependendo da mudança de escala. Mostramos também que nossa técnica pode ser útil para resolver algumas equações integrais de Volterra. Finalmente, as soluções das equações diferenciais κ-fracionárias podem ser deduzidas da representação da solução das versões Caputo ou Riemann-Liouville via mudança de escala.

Palavras-chave: Integrais Fracionárias; Derivadas Fracionárias; Logaritmo de Kaniadakis.

Abstract

In this article, more general types of fractional operators with κ-deformed logarithm kernels are proposed. We analyse the new operators and prove various facts about them, including a semigroup property. Results of existence are established in appropriate functional spaces. We prove that these results are valid at once for several standard fractional operators such as the Riemann-Liouville and Caputo operators, the Hadamard operators depending on the of the scaling function. We also show that our technique can be useful to solve a wide range of Volterra integral equations. Finally, the solutions of the κ-fractional differential equations can be deduced from the solution representation of the Caputo or Riemann-Liouville versions via scaling.

Keywords: Fractional Integrals; Fractional Derivatives; Kaniadakis Deformed Logarithm

MSC (2020): 26A33; 26A99; 34A08.

Received: 29 April 2022 / **Accepted:** 15 June 2022

1 Introduction

Fractional calculus is an emerging field of applied mathematics devoted to the use of mathematical methods and applications of integro-differential equations involving fractional operators. In literature there are many different definitions of fractional derivatives and integrals (see [1, 2, 3, 4, 5, 6] and the references therein). One of these definitions was introduced by Hadamard [7] in 1892. The Hadamard fractional derivative can be regarded as generalization of the operator $(t\frac{d}{dt})^n$, while the Riemann–Liouville derivative can be viewed as an extension of the classic differential operator $(\frac{d}{dt})^n$ cursorily [8].

The mathematical foundations of the Kaniadakis statistical mechanics are based on the κ-deformed logarithm function (or Kaniadakis logarithm) [9]: $\ln_\kappa x = \frac{x^\kappa - x^{-\kappa}}{2\kappa}$, and its inverse function, the κ-deformed exponential (or Kaniadakis exponential): $\exp_\kappa x = \exp(\frac{1}{\kappa}\operatorname{arsinh}\kappa x)$. Note that for any $|\kappa| < 1$, $\ln_\kappa$ and $\exp_\kappa(x)$ are continuous, monotonic, increasing functions, normalized in $\ln_\kappa(1) = 0$ and $\exp_\kappa(0) = 1$, with $\ln_\kappa(\mathbb{R}^+) \subseteq \mathbb{R}$ and $\exp_\kappa(\mathbb{R}) \subseteq \mathbb{R}^+$. In particular we obtain $\exp_\kappa(x)\exp_\kappa(-x) = 1$ and $\ln_\kappa(x) + \ln_\kappa(1/x) = 0$. Moreover, these κ-functions, fulfil the following scaling-laws $\exp_\kappa(\mu\, x) = \exp_{\kappa'}(x)^\mu$ and $\ln_\kappa(x^\mu) = \mu \ln_{\kappa'}(x)$, where $\kappa' = \mu\,\kappa$, (see [10], [11], [12] for details).

This paper is structured in the following manner: In Section 2, we provide some preliminaries for fractional calculus. In Section 3, we consider fractional calculus with Kaniadakis logarithm kernels and state their properties. In Section 4, we establishing appropriate function spaces in which they can be applied. In Section 5, we discuss the κ-Caputo-Hadamard derivatives. We also give some examples where Volterra integral equations of the second kind are solved. Finally, some conclusions are presented in Section 7.

2 Preliminaries and Background Materials

In this section, we present some basic notations, definitions, and preliminary results, which will be used throughout this paper. We also recall some essential results whose proofs can be seen in the literature.

Let $[a, b]$ $(0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$. Denote by $\mathcal{C}[a, b]$, the spaces of the continuous function f on $[a, b]$ with norm defined by [13]

$$\|f\|_{C[a,b]} = \max_{t\in[a,b]} |f(t)|,$$

and $AC^n[a, b]$, the space of n-times absolutely continuous differentiable functions on $[a, b]$, given by

$$AC^n[a, b] = \{\mathsf{h} : [a, b] \to \mathbb{R} : \mathsf{h}^{(n-1)} \in AC[a, b]\}.$$

Definition 2.1 ([14]). For $\in \mathbb{C}$, $\mathrm{Re}(z) > 0$, the Euler gamma function is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t.$$

Moreover, $\Gamma(z+1) = z\Gamma(z)$.

Definition 2.2 ([7]). The Riemann–Liouville (RL) fractional integral with order $\alpha > 0$ (or $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) of a given function $f \in L^1(a,b)$ is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t,$$

where $x \in (a,b)$ and $a < b$ in $\mathbb{R}$. This is the fractional power of the standard differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$.

Definition 2.3. The (left) Riemann-Liouville fractional derivative of order α and its Caputo modification are defined for any function $f \in AC^n[a,b]$ respectively by

$$^{RL}D_a^\alpha f(t) = \left(I_a^{n-\alpha} f\right)^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds \tag{1}$$

and

$$^{C}D_a^\alpha f(t) = I_a^{n-\alpha}\left(f^{(n)}\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds, \tag{2}$$

where $n := \lfloor \mathrm{Re}(\alpha) \rfloor + 1$ so that $n - 1 \le \mathrm{Re}(\alpha) < n$.

We note that the Riemann–Liouville and Caputo fractional derivatives both stem from the same definition of fractional integrals, simply combining this with the original differentiation operation in one order or the other.

Definition 2.4 ([7]). The Riemann–Liouville fractional integral with order $\alpha > 0$ (or $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) of a given function f with respect to a monotonic C^1 function g is defined as

$$_aI_{g(x)}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(g(x) - g(t)\right)^{\alpha-1} f(t) g'(t)\,\mathrm{d}t,$$

where $x \in (a,b)$ and $a < b$ in $\mathbb{R}$. This is the fractional power of the operator $\frac{\mathrm{d}}{\mathrm{d}g(x)} = \frac{1}{g'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}$ of differentiation with respect to the function g.

The Riemann–Liouville and Caputo fractional derivatives with order $\alpha > 0$ (or $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \ge 0$) of a given function f, with respect to a monotonic C^1 function g, are defined respectively as

$$^R_aD_{g(x)}^\alpha f(x) = \left(\frac{1}{g'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n {_aI_{g(x)}^{n-\alpha}} f(x),$$

$$^C_aD_{g(x)}^\alpha f(x) = {_aI_{g(x)}^{n-\alpha}} \left(\frac{1}{g'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x),$$

where $n-1 \leq \operatorname{Re}(\alpha) < n \in \mathbb{Z}^{+}$, $x \in (a,b)$, and $a < b$ in $\mathbb{R}$.

Note that in the particular case $g(x) = x$, ${}_aI^{\alpha}_{g(x)}$ reduces to Riemann-Liouville fractional integral of order α. In the case $g(x) = \ln x$ $(a > 0)$, ${}_aI^{\alpha}_{g(x)}$ reduces to Hadamard fractional integral of order α (see [15], [16], [17] for more details).

3 Fractional operators with κ-logarithm kernel

In this section, we shall exploit the concept of our new Hadamard type fractional integral operator.

The n^{th}, $n \in \mathbb{Z}^{+}$ order fractional integral of a function f with respect to the parametric function $\int_1^x \frac{1}{g_\kappa(s)}\,\mathrm{d}s$, with $g_\kappa(x) = \frac{2x}{x^{\kappa}+x^{-\kappa}}$, has the form

$$\begin{aligned} {}_a\mathbb{I}^{n}_{\kappa} f(x) &= \int_a^x \frac{\mathrm{d}\tau_1}{g_\kappa(\tau_1)} \int_a^{\tau_1} \frac{\mathrm{d}\tau_2}{g_\kappa(\tau_2)} \cdots \int_a^{\tau_{n-1}} f(\tau)\, \frac{\mathrm{d}\tau}{g_\kappa(\tau)} \\ &= \frac{1}{(n-1)!} \int_a^x \left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-1} f(\tau) \frac{\mathrm{d}\tau}{g_\kappa(\tau)}, \quad \kappa \in (0,1),\ x > a. \end{aligned} \tag{3}$$

Remark 3.1. The fractional integral in (3) coincides with the Hadamard fractional integral when $\kappa = 0$.

The corresponding derivative is

$$\begin{aligned} \left(\delta^1_\kappa f\right)(x) &= \left(g_\kappa(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(f(x)\right), \\ \left(\delta^n_\kappa f\right)(x) &= \delta^1_\kappa \left(\delta^{n-1}_\kappa f\right)(x) \\ &= \left(g_\kappa(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(f(x)\right). \end{aligned} \tag{4}$$

Remark 3.2. We can also give a definition of limit form of δ_κ-derivative operator in the following way,

$$\left(g_\kappa(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\ln \frac{g_\kappa(x)+h}{g_\kappa(x)}}. \tag{5}$$

The fractional versions of the integral in (3) and the derivative (in Riemann-Liouville settings) in (4) are

$$\left({}_{a^+}\mathbb{I}^{\alpha}_{\kappa} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-1} f(\tau) \frac{\mathrm{d}\tau}{g_\kappa(\tau)}, \quad \kappa \in (0,1),\ x > a \tag{6}$$

and

$$\begin{aligned} \left({}_{a^+}\mathbb{D}_\kappa^\alpha f\right)(x) &= \left(\delta_\kappa^n \, \mathbb{I}_\kappa^{n-\alpha} f\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \delta_\kappa^n \left(\int_a^x \left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-1} f(\tau) \frac{\mathrm{d}\tau}{g_\kappa(\tau)} \right), \end{aligned} \tag{7}$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ being the integer part of α.

Property 3.3. Let $n \in \mathbb{Z}^+$, $\alpha > 0$ and $\kappa \in (0,1)$,

1. $\left(\delta_\kappa^n \, {}_{a^+}\mathbb{I}_\kappa^n \, f\right)(x) = f(x)$.
2. $\left({}_{a^+}\mathbb{I}_\kappa^n \, \delta_\kappa^n f\right)(x) = f(x) - \sum_{j=0}^{n-1} \frac{(\ln_\kappa x - \ln_\kappa a)^j}{j!} \delta_\kappa^j f(a)$.

Proof. Item (1) can be easily proved using (3), (4) and the Leibniz rule for integrals. For (2), by

$$\begin{aligned} \left({}_{a^+}\mathbb{I}_\kappa^n \, \delta_\kappa^n f\right)(x) &= \frac{1}{(n-1)!} \int_a^x \left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-1} \delta_\kappa^n f(\tau) \frac{\mathrm{d}\tau}{g_\kappa(\tau)} \\ &= \frac{1}{(n-1)!} \int_a^x \left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-1} \frac{\mathrm{d}}{\mathrm{d}\tau} \delta_\kappa^{n-1} f(\tau) \, \mathrm{d}\tau. \end{aligned} \tag{8}$$

Using integration by parts, we deduce

$$\left({}_{a^+}\mathbb{I}_\kappa^n \, \delta_\kappa^n f\right)(x) = -\frac{\left(\ln_\kappa x - \ln_\kappa a\right)^{n-1}}{(n-1)!} \delta_\kappa^{n-1} f(a) + \int_a^x \frac{\left(\ln_\kappa x - \ln_\kappa \tau\right)^{n-2}}{(n-2)!} \delta_\kappa^{n-1} f(\tau) \mathrm{d}\tau \tag{9}$$

Repeating the same procedure $n-2$ times, we arrive at item (2). □

Lemma 3.4. *1. For $\alpha > 0$ and $\beta > 0$, we have*

$$\left({}_{a^+}\mathbb{I}_\kappa^\alpha (\ln_\kappa t - \ln_\kappa a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln_\kappa t - \ln_\kappa a)^{\beta+\alpha-1}. \tag{10}$$

2. For $\alpha < n$ and $\beta > 0$, we have

$$\left({}_{a^+}\mathbb{D}_\kappa^\alpha (\ln_\kappa t - \ln_\kappa a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln_\kappa t - \ln_\kappa a)^{\beta-\alpha-1}. \tag{11}$$

Proof. With the change of variables $z = \frac{\ln_\kappa \tau - \ln_\kappa a}{\ln_\kappa x - \ln_\kappa a}$ and with the help of the Beta function

$$B(x,y) = \int_0^1 z^{u-1}(1-z)^{1-v} \, \mathrm{d}z = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad x, y > 0,$$

we obtain

$$\begin{aligned}\left({}_{a^+}\mathbb{I}^n_\kappa(\ln_\kappa t-\ln_\kappa a)^{\beta-1}\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x(\ln_\kappa x-\ln_\kappa\tau)^{n-1}(\ln_\kappa\tau-\ln_\kappa a)^{\beta-1}\frac{\mathrm{d}\tau}{g_\kappa(\tau)}\\ &= \frac{(\ln_\kappa x-\ln_\kappa a)^{\beta+\alpha-1}}{\Gamma(\alpha)}\int_0^1 z^{\beta-1}(1-z)^{\alpha-1}\,\mathrm{d}z\\ &= \frac{(\ln_\kappa x-\ln_\kappa a)^{\beta+\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}\\ &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln_\kappa x-\ln_\kappa a)^{\beta+\alpha-1}. \qquad (12)\end{aligned}$$

The second formula is proven is a similar way. □

Lemma 3.5. *Let $n\in\mathbb{Z}^+$ and let f be a continuous function, n times derivable in $[a,b]$. Then*

$$\delta^n_\kappa f=(f\circ\exp_\kappa)^{(n)}\circ\ln_\kappa \quad \textit{in } [a,b]. \qquad (13)$$

Proof. We prove the result by induction. The case $n=1$ is straightforward. Suppose (13) is satisfied, then

$$\begin{aligned}(f\circ\exp_\kappa)^{(n+1)}\circ\ln_\kappa &= ((\delta^n_\kappa f)\circ\exp_\kappa)'\circ\ln_\kappa\\ &= \left[\left(\frac{1}{\ln'_\kappa}(\delta^n_\kappa f)'\right)\circ\exp_\kappa\right]\circ\ln_\kappa\\ &= \delta^{n+1}_\kappa f.\end{aligned}$$

□

Let $1\le p<\infty$ and define the space of p-integrable functions with respect to the Kaniadakis logarithm:

$$L^p_\kappa(a,b):=\left\{f:[a,b]\to\mathbb{C},\ \int_a^b|f(s)|^p\,\mathrm{d}\ln_\kappa(s)<\infty\right\}. \qquad (14)$$

Remark 3.6. 1. If $1/g_\kappa(x)$ is bounded on $[a,b]$, then $L^p_\kappa(a,b)=L^p(a,b)$.

2. $\ln_\kappa$ is invertible and $\exp_k\in C^n(\bar{a},\bar{b}]$, with the notation $\bar{\xi}=\ln_\kappa(\xi)$.

Corolary 3.7. *Let $1\le p<\infty$ and $n\in\mathbb{Z}^+$. Then*

$$f\in L^p_\kappa(a,b)\Longleftrightarrow f\circ\exp_\kappa\in L^p(\bar{a},\bar{b}),$$

$$f\in AC^n_\kappa[a,b]\Longleftrightarrow f\circ\exp_\kappa\in AC^n[\bar{a},\bar{b}].$$

Proof. The first assertion is immediate. Denote $z = f \circ \exp_\kappa$, then we obtain by Lemma 3.5

$$\begin{aligned} z \in AC^n[\bar{a},\bar{b}] &\Leftrightarrow \exists\, c \in \mathbb{R} \text{ and } \varphi \in L^1(\bar{a},\bar{b}) \text{ s.t. } z^{(n-1)}(\bar{x}) = c + \int_{\bar{a}}^{\bar{x}} \varphi(s)\,\mathrm{d}s, \quad \forall\, \bar{x} \in [\bar{a},\bar{b}] \\ &\Leftrightarrow z^{(n-1)} \circ \ln_\kappa x = c + \int_{\ln_\kappa a}^{\ln_\kappa x} \varphi(s)\,\mathrm{d}s, \quad \forall\, x \in [a,b] \\ &\Leftrightarrow \delta_\kappa^{n-1} f(x) = c + \int_{\ln_\kappa a}^{\ln_\kappa x} \varphi(s)\,\mathrm{d}s, \quad \forall\, x \in [a,b] \\ &\Leftrightarrow \delta_\kappa^{n-1} f(x) = c + \int_a^x (\ln_\kappa' s)\, \varphi \circ \ln_\kappa(s)\,\mathrm{d}s, \quad \forall\, x \in [a,b] \\ &\Leftrightarrow f \in AC_\kappa^n[a,b]. \end{aligned}$$

□

Theorem 3.8. *Let* $\alpha \geq 0$ *and* $n = [\alpha] + 1$*. Then for any* $u \in L_\kappa^p(a,b)$, $1 \leq p < \infty$, *we have*

i) ${}_{a^+}\mathbb{I}_\kappa^\alpha f = \left(I_{\bar{a}}^\alpha \left(f \circ \exp_\kappa\right)\right) \circ \ln_\kappa, \ \forall f \in AC_\kappa^n[a,b]$,

ii) ${}_{a^+}\mathbb{D}_\kappa^\alpha f = \left({}^{RL}D_{\bar{a}}^\alpha \left(f \circ \exp_\kappa\right)\right) \circ \ln_\kappa$,

Proof. Let $f \in L_\kappa^p(a,b)$. Then, using Corollary 3.7, we have $f \circ \exp_\kappa \in L^p(\bar{a},\bar{b})$ and hence $I_{\bar{a}}^\alpha\left(f \circ \exp_\kappa\right)$ is well defined on $[\bar{a},\bar{b}]$. It follows that for a.e. $x \in [a,b]$

$$ {}_{a^+}\mathbb{I}_\kappa^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{\bar{a}}^{\bar{x}} (\ln_\kappa x - \xi)^{1-\alpha} f(\exp_\kappa(\xi))\,\mathrm{d}\xi = I_{\bar{a}}^\alpha\left(f \circ \exp_\kappa\right)(\ln_\kappa x). $$

Now, let $f \in AC_\kappa^n[a,b]$. Then, we have by Corollary 3.7 that $f \circ \exp_\kappa \in AC[a,b]$ and hence ${}^{RL}D_{\bar{a}}^\alpha\left(f \circ \exp_\kappa\right)$ is well defined on $[\bar{a},\bar{b}]$. It follows by $i)$ and Lemma 3.5 that for a.e. $x \in [a,b]$

$$\begin{aligned} {}_{a^+}\mathbb{D}_\kappa^\alpha f(x) &= \left(\delta_\kappa^n\, \mathbb{I}_\kappa^{n-\alpha} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left[\left(\mathbb{I}_\kappa^{n-\alpha} f\right) \circ \exp_\kappa\right](\ln_\kappa x) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left[I_{\bar{a}}^{n-\alpha}\left(f \circ \exp_\kappa\right)\right](\ln_\kappa x) \\ &= {}^{RL}D_{\bar{a}}^\alpha\left(f \circ \exp_\kappa\right)(\ln_\kappa x). \end{aligned}$$

□

4 Boundedness in the space $L_\kappa^p(a,b)$

In this section, we define the space where the with respect to the Kaniadakis logarithm are bounded and present some properties of these operators.

Theorem 4.1. *Let* $p \geq 1$*. Then, the fractional integral operator with respect to the Kaniadakis logarithm* ${}_{a^+}\mathbb{I}_\kappa^\alpha$ *is bounded in* $L_\kappa^p(a,b)$*:*

$$ \|{}_{a^+}\mathbb{I}_\kappa^\alpha f\|_{L_\kappa^p(a,b)} \leq K_\kappa \|f\|_{L_\kappa^p(a,b)} \quad \textit{with } K_\kappa = \frac{(\bar{b}-\bar{a})^\alpha}{\Gamma(\alpha+1)}. $$

Proof. First, remark that $\|f \circ \ln_\kappa\|_{L^p_\kappa(a,b)} = \|f\|_{L^p(\bar{a},\bar{b})}$, with the notation $\bar{\xi} = \psi(\xi)$. Using Proposition 3.8 and the continuity of the operator I^α_a,

$$\|I^\alpha_a f\|_{L^p(a,b)} \leq K\|f\|_{L^p(a,b)} \quad \text{with } K = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)},$$

one obtain

$$\begin{aligned}\|{}_{a^+}\mathbb{I}^\alpha_\kappa f\|_{L^p_\kappa(a,b)} &= \|(I^\alpha_{\bar{a}}(f \circ \exp_\kappa)) \circ \ln_\kappa\|_{L^p_\kappa(a,b)} \\ &= \|I^\alpha_{\bar{a}}(f \circ \exp_\kappa)\|_{L^p(\bar{a},\bar{b})} \\ &\leq K_\kappa \|f \circ \exp_\kappa\|_{L^p(\bar{a},\bar{b})} \\ &= K_\kappa \|f\|_{L^p_\kappa(a,b)}.\end{aligned}$$

□

Theorem 4.2. (**semi-group law**) *Let $\alpha > 0$ and $\beta > 0$. Then, for any $1 \leq p < \infty$ and $u \in L^p_\psi(a,b)$*

1. ${}_{a^+}\mathbb{I}^\alpha_\kappa\, {}_{a^+}\mathbb{I}^\beta_\kappa f = {}_{a^+}\mathbb{I}^{\alpha+\beta}_\kappa f.$
2. ${}_{a^+}\mathbb{D}^\alpha_\kappa\, {}_{a^+}\mathbb{I}^\alpha_\kappa f = f.$
3. ${}_{a^+}\mathbb{D}^\beta_\kappa {}_{a^+}\mathbb{I}^\alpha_\kappa f = {}_{a^+}\mathbb{I}^{\alpha-\beta,\psi}_\kappa f \quad \forall\ \alpha \geq \beta.$
4. *Let $m \in \mathbb{Z}^+$, then*

$$\delta^m_\kappa\, {}_{a^+}\mathbb{I}^\alpha_\kappa f = \begin{cases} {}_{a^+}\mathbb{I}^{\alpha-m}_\kappa f & \text{if } m \leq \alpha \\ {}_{a^+}\mathbb{D}^{m-\alpha}_\kappa f & \text{if } m \geq \alpha. \end{cases}$$

Proof. We only prove the first assertion, since the proofs of the other identities follow a similar idea. Let $f \in L^p_\kappa(a,b)$, then $f \circ \exp_\kappa \in L^p(\bar{a},\bar{b})$. It follows from Lemma 2.3 in [7] and Proposition 3.8 that for a.e. $x \in [a,b]$

$$\begin{aligned}{}_{a^+}\mathbb{I}^\alpha_\kappa\, {}_{a^+}\mathbb{I}^\beta_\kappa f(x) &= I^\alpha_{\bar{a}}\left(\left({}_{a^+}\mathbb{I}^\beta_\kappa f\right) \circ \exp_\kappa\right)(\ln_\kappa x) \\ &= I^\alpha_{\bar{a}}\left(I^\beta_{\bar{a}}(f \circ \exp_\kappa)\right)(\ln_\kappa x) \\ &= I^{\alpha+\beta}_{\bar{a}}(f \circ \exp_\kappa)(\ln_\kappa x) \\ &= {}_{a^+}\mathbb{I}^{\alpha+\beta}_\kappa f(x).\end{aligned}$$

□

5 Caputo type fractional derivative operator

Definition 5.1. Let $\alpha > 0$ and $n = [\alpha] + 1$. The κ-Caputo fractional derivative of order α of $f \in \mathcal{C}^n[a,b]$ is defined by

$$\begin{aligned} {}^{C}_{a}\mathbb{D}^{\alpha}_{\kappa}\, y(t) &= {}_{a^+}\mathbb{I}^{n-\alpha}_{\kappa}\, (\delta^n_\kappa y)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(\delta^n_\kappa y)(s)}{(\ln_\kappa t - \ln_\kappa s)^{\alpha-n+1}} \frac{\mathrm{d}s}{g_\kappa(s)}. \end{aligned} \tag{15}$$

Define the space

$$\mathcal{AC}^n_\kappa[a,b] := \left\{ y : [a,b] \to \mathbb{C} \ \text{ s.t. } \delta^{n-1}_\kappa y \in \mathcal{AC}[a,b] \right\}.$$

Lemma 5.2. *Let $n \in \mathbb{Z}^+$. Then we have the following embedding*

$$\mathcal{C}^n[a,b] \subset \mathcal{AC}^n_\kappa[a,b] \subset \mathcal{C}^{n-1}[a,b] \subset \cdots \subset \mathcal{C}^1[a,b] \subset \mathcal{AC}_\kappa[a,b] \subset \mathcal{C}[a,b]$$

where $\mathcal{C}^n[a,b]$ denotes the set of continuously differentiable functions up to order n.

Proof. The proof is direct. □

Theorem 5.3. *Let $y \in \mathcal{AC}^n_\kappa[a,b]$. Then for a.e. $t \in [a,b]$*

$${}^{C}_{a}\mathbb{D}^{\alpha}_{\kappa}\, y(t) = {}_{a^+}\mathbb{I}^{n-\alpha}_{\kappa}\, (\delta^n_\kappa y)(t).$$

In particular, ${}^{C}_{a}\mathbb{D}^{n}_{\kappa}\, y = \delta^n_\kappa u$ for any $n \in \mathbb{Z}^+$.

Proof. Using Corollary 3.7, we have $y \circ \exp_\kappa \in \mathcal{AC}^n[\bar{a}, \bar{b}]$. It follows from [7, Theorem 2.1], Proposition 3.8 and Lemma 3.5 that for a.e. $t \in [a,b]$

$$\begin{aligned} {}^{C}_{a}\mathbb{D}^{\alpha}_{\kappa}\, y(t) &= {}^{C}D^{\alpha}_{\bar{a}}\, (y \circ \exp_\kappa)(\ln_\kappa t) \\ &= I^{n-\alpha}_{\bar{a}} \left((y \circ \exp_\kappa)^{(n)} \right) (\ln_\kappa t) \\ &= I^{n-\alpha}_{\bar{a}} \left((\delta^n_\kappa y) \circ \exp_\kappa \right) (\ln_\kappa t) \\ &= {}_{a^+}\mathbb{I}^{n-\alpha}_{\kappa}\, (\delta^n_\kappa y)(t). \end{aligned}$$

□

Theorem 5.4. *(**Composition**) Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. Then, for $u \in C[a,b]$*

$${}^{C}_{a}\mathbb{D}^{\alpha}_{\kappa}\, a \left(I^{\alpha,\psi}_{a} y \right)(t) = y(t),$$

and for $y \in AC^n_\kappa[a,b]$

$$ {}_{a^+}\mathbb{I}^\alpha_\kappa \left({}^C_a\mathbb{D}^\alpha_\kappa\, y\right)(t) = y(t) - \sum_{j=0}^{n-1} \frac{\left(\delta^j_\kappa y\right)(a)}{j!}\Big(\ln_\kappa t - \ln_\kappa a\Big)^j . $$

Proof. Since ψ is continuous, then $y \circ \exp_\kappa$ is also continuous. The rest of the proof is an consequence of Theorem 3.8, Lemma 3.5 and Corollary 3.7. □

Theorem 5.5. ${}^C_a\mathbb{D}^\alpha_\kappa\, y \in \mathcal{C}[a,b]$*. In particular, if* $\alpha \notin \mathbb{Z}^+$ *then* ${}^C_a\mathbb{D}^\alpha_\kappa\, y(a) = 0$*.*

Proof. Since $\exp_\kappa \in \mathcal{C}^n[\bar{a},\bar{b}]$ then $y \circ \exp_\kappa \in \mathcal{C}^n[\bar{a},\bar{b}]$. Using that $\mathcal{C}^n[a,b] \subset AC^n_\kappa[a,b]$, we obtain by Theorem 2.2 in [7] and Proposition 3.8 that ${}^C_a\mathbb{D}^\alpha_\kappa\, y = \left({}^C D^\alpha_{\bar{a}}\left(y \circ \exp_\kappa\right)\right) \circ \ln_\kappa \in \mathcal{C}[a,b]$. Moreover, we have from Theorem 2.2 in [7] that ${}^C D^\alpha_{\bar{a}}\left(y \circ \exp_\kappa\right)(\bar{a}) = 0$ if $\alpha \notin \mathbb{Z}^+$. Applying again Proposition 3.8 yields ${}^C_a\mathbb{D}^\alpha_\kappa\, y(a) = 0$. □

Lemma 5.6. *Let* $\alpha > 0$*. Then*

$$ {}^C_a\mathbb{D}^\alpha_\kappa \Big(\ln_\kappa t - \ln_\kappa a\Big)^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}\Big(\ln_\kappa t - \ln_\kappa a\Big)^{\beta-\alpha}, \qquad \beta > n-1, $$

$$ {}^C_a\mathbb{D}^\alpha_\kappa\, a\Big(\ln_\kappa t - \ln_\kappa a\Big)^k = 0, \qquad k = 0, 1, \ldots, n-1. $$

Proof. A direct consequence of Theorem 3.8. □

6 Fractional differential equations

Let $\alpha > 0$ and $n = [\alpha] + 1$. Consider the following fractional differential system

$$ \begin{cases} {}_{a^+}\mathbb{D}^\alpha_\kappa\, y(t) = f(t, y(t)), \quad t \in [a,b] \\ \\ {}_{a^+}\mathbb{D}^{\alpha-k}_\kappa\, y(a) = a_k, \quad k = 1, \ldots, n-1, \quad \lim\limits_{t\to a} {}_{a^+}\mathbb{I}^{n-\alpha}_\kappa\, y(t) = a_n. \end{cases} \tag{16} $$

with f a given function and $a_k \in \mathbb{R}$ for $k = 1, \ldots, n$. We have the following integral representation of the solution of (16).

Theorem 6.1. *Let* $U \subseteq \mathbb{R}$ *be an open set and assume* $f : (a,b] \times U \to \mathbb{R}$ *is a function such that* $t \mapsto f(t,\cdot) \in L^1_\kappa(a,b)$*. Then a function* $y \in L^1_\kappa(a,b)$ *is a solution of* (16) *if and only if* y *is a solution of the non-linear second kind Volterra integral equation*

$$ y(t) = \sum_{j=1}^{n} \frac{a_j}{\Gamma(\alpha-j+1)}\Big(\ln_\kappa t - \ln_\kappa a\Big)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s,y(s))}{\left(\ln_\kappa t - \ln_\kappa s\right)^{1-\alpha}} \frac{\mathrm{d}s}{g_\kappa(s)} \tag{17} $$

with $a_j = {}_{a^+}\mathbb{D}^{\alpha-k}_\kappa\, y(a)$ *for* $j = 1, \ldots, n-1$*, and* $a_n = \lim\limits_{t\to a} {}_{a^+}\mathbb{I}^{n-\alpha}_\kappa\, y(t)$*.*

Proof. Using Proposition 3.8, one can show that y is a solution of (16) if and only if $z = y \circ \exp_\kappa$ is a solution of the system

$$\begin{cases} {}^{RL}D^{\alpha}_{\bar{a}} z(x) = F(x, z(x)), \quad x \in [\bar{a}, \bar{b}] \\ \\ {}^{RL}D^{\alpha-k}_{\bar{a}} z(\bar{a}) = a_k, \quad k = 1, \dots, n-1, \quad \lim\limits_{x \to \bar{a}} I^{n-\alpha}_{\bar{a}} z(x) = a_n. \end{cases} \tag{18}$$

with $F(x, y) = f(\exp_\kappa x, y)$. Noticing that $t \mapsto f(t, \cdot) \in L^1_\psi(a, b) \Leftrightarrow x \mapsto F(x, \cdot) \in L^1(\bar{a}, \bar{b})$ and $u \in L^1_\psi(a, b) \Leftrightarrow z = y \circ \exp_\kappa \in L^1(\bar{a}, \bar{b})$, and using Theorem 3.1 in [7], we deduce that z is a solution of (18) if and only if z satisfies for a.e. $x \in [\bar{a}, \bar{b}]$

$$z(x) = \sum_{j=1}^{n} \frac{a_j}{\Gamma(\alpha - j + 1)} \Big(x - \bar{a}\Big)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_{\bar{a}}^{x} \frac{F(s, z(s))}{(x - s)^{1-\alpha}} \, ds. \tag{19}$$

Finally, the result follows by taking $x = \ln_\kappa t$ in equation (19). □

Define the space

$$L^{\alpha}_{\kappa}(a, b) := \Big\{ \varphi \in L^1_\kappa(a, b), \ {}_{a^+}\mathbb{D}^{\alpha}_{\kappa} \varphi \in L^1_\kappa(a, b) \Big\} \tag{20}$$

where $L^1_\kappa(a, b)$ is given in (14). Then we have the following result.

Theorem 6.2. *Let $U \subseteq \mathbb{R}$ be an open set and let $f : (a, b] \times U \to \mathbb{R}$ be a function such that $t \mapsto f(t, \cdot) \in L^1_\kappa(a, b)$. Assume that f fulfills a Lipschitz condition with respect to its second variable. Then the Cauchy problem (16) admits a unique solution $y \in L^{\alpha}_{\kappa}(a, b)$.*

Proof. We have established in Proposition 6.1 that y is a solution of (16) if and only if $z := y \circ \exp_\kappa$ is a solution of (18). Since $(t, y) \mapsto F(t, y) := f(\exp_\kappa t, y)$ is Lipschitzian with respect to its second variable, we obtain from Theorem 3.3 in [7] that the system (18) admits a unique solution $z \in L^{\alpha}(\bar{a}, \bar{b}) := \Big\{ \varphi \in L^1(\bar{a}, \bar{b}), \ {}^{RL}D^{\alpha}_{\bar{a}} \varphi \in L^1(\bar{a}, \bar{b}) \Big\}$. Finally, the result follows by noticing that $z \in L^{\alpha}(\bar{a}, \bar{b}) \Leftrightarrow y = z \circ \ln_\kappa \in L^{\alpha}_{\psi}(a, b)$. □

Corolary 6.3. *A function $u \in L^1_\kappa(a, b)$ is a solution of (16) if and only if $y = z \circ \ln_\kappa$ with $z \in L^1(\bar{a}, \bar{b})$ is a solution of the Riemann-Liouville fractional differential system*

$$\begin{cases} {}^{RL}D^{\alpha}_{\bar{a}} z(t) = F(t, z(t)), \quad t \in [\bar{a}, \bar{b}] \\ \\ {}^{RL}D^{\alpha-j}_{\bar{a}} z(\bar{a}) = a_j, \quad j = 1, \dots, n-1, \quad \lim\limits_{t \to \bar{a}} I^{n-\alpha}_{\bar{a}} z(t) = a_n. \end{cases}$$

with the notation $\bar{\xi} = \psi(\xi)$ and $F(t, x) = f(\psi^{-1}(t), x)$.

Let $\alpha > 0$. Consider the following κ-Caputo fractional differential system with respect to another function

$$\begin{cases} {}^{C}_{a}\mathbb{D}^{\alpha}_{\kappa}\, y(t) = f(t, y(t)), \quad t \in [a, b] \\ \\ \left(\delta^{j} y\right)(a) = a_j, \quad j = 0,\, 1, \ldots, n-1 \end{cases} \tag{21}$$

with f a given function and $a_j \in \mathbb{R}$ for $j = 0,\, 1, \ldots, n-1$.

Theorem 6.4. *A function $y \in AC^n_\kappa[a, b]$ is a solution of (21) if and only if $y = z \circ \ln_\kappa$ with $z \in AC^n[\bar{a}, \bar{b}]$ is a solution of the κ-Caputo differential system*

$$\begin{cases} {}^{C}D^{\alpha}_{\bar{a}} z(t) = F(t, z(t)), \quad t \in [\bar{a}, \bar{b}] \\ \\ z^{(j)}(\bar{a}) = a_j, \quad j = 0,\, 1, \ldots, n-1 \end{cases} \tag{22}$$

with $F(t, x) = f(\exp_\kappa t, x)$.

Proof. A direct consequence of Theorem 3.8, Lemma 3.5 and Corollary 3.7. □

Theorem 6.5. *Assume f is continuous over $[a, b] \times \mathbb{R}$. Then a function $y \in AC^n_\psi[a, b]$ is a solution of (21) if and only if y is a solution of the of the non-linear second kind Volterra integral equation*

$$y(t) = \sum_{j=0}^{n-1} \frac{a_j}{j!}\Big(\ln_\kappa t - \ln_\kappa a\Big)^j + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s, y(s))}{\left(\ln_\kappa t - \ln_\kappa s\right)^{1-\alpha}} \frac{\mathrm{d}s}{g_\kappa(s)} \tag{23}$$

with $a_j = \left(\delta^j_\kappa y\right)(a)$ for $j = 0,\, 1, \ldots, n-1$.

Proof. The proof can be implemented by using Theorems 5.4 and 4.2, and Lemma 5.6. □

7 Concluding remark

The paper presents new fractional integral operators containing Kaniadakis logarithm functions in their kernels. We already know that we can deduce Hadamard integrals for the special cases of κ. We show that a Cauchy problem is equivalent to the Volterra integral equation of the second kind. The mathematical analysis for the solutions of κ-Caputo fractional differential equations can be deduced directly from their Caputo counterparts. The existence of new generalized integral operators may be useful in several applications of fractional calculus in science and engineering.

References

[1] E. C. De Oliveira and J. A. Tenreiro Machado, "A review of definitions for fractional derivatives and integral," *Mathematical Problems in Engineering*, vol. 2014, 2014. https://doi.org/10.1155/2014/238459

[2] G. Sales Teodoro, J. A. Tenreiro Machado and E. Capelas de Oliveiraa, "A review of definitions of fractional derivatives and other operators," *Journal of Computational Physics*, vol. 388, pp. 195-208, 2019.

[3] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," Journal of Computational and Applied Mathematics, vol. 346, pp. 378-389, 2019.

[4] F. S. Silva, "Conformable Fractional Integral Equations of the Second Kind," Mathematica Aeterna; vol. 8, no.4, pp. 199-205, 2018.

[5] C. de A. S. Reis and R. V. da Silva Junior, "A new non-conformable derivative based on Tsallis's q-exponential function," INTERMATHS, vol. 2, no. 2, pp. 106-118, 2021.

[6] F. S. Silva, D. M. Moreira, and M. A. Moret, "Conformable Laplace Transform of Fractional Differential Equations," Axioms, vol. 7, no. 3, p. 55, 2018.

[7] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*. North Holland Mathematics Studies, 204, Amsterdam , 2006.

[8] LI MA and CHANGPIN LI, "On Hadamard fractional calculus," *Fractals*, vol. 25, no. 03, p. 1750033, 2017.

[9] G. Kaniadakis, "Non-linear kinetics underlying generalized statistics," *Physica A: Statistical mechanics and its applications*, vol. 296, no. 3-4, pp. 405-425, 2001.

[10] G. Kaniadakis and A. M. Scarfone, "A new one-parameter deformation of the exponential function," *Physica A: Statistical Mechanics and its Applications*, vol. 305, no. 1-2, pp. 69-75, 2002.

[11] J. Naudts, *Generalised thermostatistics*. Springer Science & Business Media, 2011.

[12] G. Kaniadakis, "Theoretical Foundations and Mathematical Formalism of the Power-Law Tailed Statistical Distributions," Entropy, vol. 15, no. 12, pp. 3983-4010, 2013.

[13] R. Magnus, *Fundamental Mathematical Analysis*. Springer, Cham, Switzerland, 2020.

[14] G. E. Andrews, R. Askey and R. Roy, *Special Functions*. Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, 1999.

[15] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460-481, 2017.

[16] F. Jarad, T. Abdeljawad and K. Shah, "On the weighted fractional operators of a function with respect to another function", *Fractals*, vol. 28, no. 08, p. 2040011, 2020.

[17] A. Fernandez and H. M. Fahad, "Weighted Fractional Calculus: A General Class of Operators," *Fractal and Fractional*, vol. 6, no. 4, p. 208, Apr. 2022.

BRIEF BIOGRAPHY

Ana Paula Perovano https://orcid.org/0000-0002-0893-8082

Doutoranda em Educação Matemática (UNESP). Mestre em Educação Matemática (PUC-SP). Professora da Universidade Estadual do Sudoeste da Bahia (UESB). Integrante do Grupo de Pesquisa TeorEMa - Interlocuções entre Geometria e Educação Matemática. Desenvolve estudos e pesquisas sobre Formação de Professores, Livros Didáticos de Matemática e Processos de Ensino e de Aprendizagem da Matemática.

Fernando Santos Silva https://orcid.org/0000-0002-2059-4550

Doutor em Modelagem Computacional e Tecnologia Industrial pelo Centro Universitário SENAI CIMATEC (Brasil). Pesquisa e desenvolve estudos sobre Cálculo Fracionário, Análise Funcional e Equações Integrodiferenciais Fracionárias.