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ARTICLE

A note on Pascal's triangle and division by eleven

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Resumo: A divisibilidade é um tema antigo que até hoje intriga e fascina pesquisadores e estudiosos. Várias regras são bem conhecidas em particular a divisibilidade por onze, uma vez que, por exemplo, um palíndromo com um número par de dígitos é divisível por onze. Nos tempos atuais, a divisibilidade tem suas aplicações, por exemplo, em criptografia. Aqui, neste artigo, mostraremos que aplicando dois procedimentos um tanto intuitivos às linhas do triângulo de Pascal deve sempre retornar números divisíveis por onze. Exemplos ilustrativos são apresentados.

Palavras-chave: Triângulo de Pascal; Divisibilidade; Palíndromos; Relação de Stifel; Teorema do binômio.

Abstract: Divisibility is an old topic that to this day intrigues and fascinates researchers and scholars. Several rules are well-known in particular the divisibility by eleven, since, for example, a palindrome with an even number of digits is divisible by eleven. In current times, divisibility has its applications, for example, in cryptography. Here, in this paper we will show that applying two somewhat intuitive procedures to the lines of Pascal's triangle shall always yield numbers divisible by eleven. Illustrative examples are presented.

Keywords: Pascal's triangle; Divisibility; Palindromes; Stifel's relationship; Binomial theorem.

Classification MSC: 05A10; 11B25

1 Introduction

Divisibility is a vast topic within number theory and has been a subject of study since remote times. Even though its early introduction in formal education (during elementary school) it plays central role in rather sophisticated fields, such as cryptography.

As divisibility is well-known and has plenty study material available, we will limit ourselves to mention two graduation level references: Gardner [1] in which divisibility rules from two to twelve are discussed; also the book Richmond & Richmond [2] which presents many divisibility criteria and some of their applications.

The Pascal's triangle [1623 – Blaise Pascal – 1662] has been fascinating generations of mathematicians. It is a fairly simple representation of binomial numbers, but its lines and columns provide all sorts of interesting relations, we mention eleven division [3],[4] and Pythagorean triples [5]. Students are firstly presented this triangle in high school, but it is rare to study its properties in depth, which happens only in some graduation

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courses.

Although Pascal's triangle is not as well-known as divisibility to the general public, it is of great importance to many fields, e.g. combinatorial analysis. It can be defined as an infinite lower triangular matrix in which the n-th line is composed by the coefficients of the Newton's binomial theorem [1643 – Isaac Newton – 1727]. Other way to define Pascal's triangle is by construction using Pascal's rule, also known as Stifel's relation [1487 – Michael Stifel – 1567], which is an identity involving binomial coefficients. For more details on the topic we suggest: Merris [6] for its quality and for being easily accessible on the internet; and the recent work of Wallis & George [7] for it presents various application on the matter.

On the present work we will focus on the divisibility by eleven and its relation to the lines of Pascal's triangle. To do so we introduce some procedures that applied to the lines yield multiples of eleven. Illustrative examples are provided. As a step towards the demonstration of the main result we also show that the reverse[i] of a number divisible by eleven is also divisible by eleven.

This article is organized as follows: in section two we introduce some preliminary information and context, highlighting the divisible by eleven rule, binomial coefficients, the binomial theorem, the Stifel's relation and the Pascal's triangle. Section three is dedicated to state and prove the main proposition.

2 Preliminaries

In order to reach our goal, first, we present some definitions and hypotheses that will be used to obtain the main results.

Lemma 2.1. Divisibility by eleven. *Let S_e and S_o be, starting from the right respectively, the sum of the digits in even and odd positions of an integer number N. N is divisible by eleven if, and only if, $|S_o - S_e|$ is divisible by eleven.*

Definition 2.2. Binomial coefficient. Let n and k be non-negative integer numbers with $k \leq n$. The binomial coefficient is given by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

where $\ell!$, with $\ell \in \mathbb{N}$, is the factorial. For $n < k$ the binomial coefficient is zero.

This binomial coefficient is interpreted as the number of subsets of k elements from a set of n elements or, in other words, the number of ways one can choose k different elements out of n distinct elements.

Theorem 2.3. Binomial theorem. *Let a and b be real numbers and $n \in \mathbb{N}$. The binomial theorem is expressed as*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \,.$$

[i] For reverse of a number we mean read it backwards. Not to be confused with the inverse of a number, which is one divided by the number.

Definition 2.4. STIFEL'S RELATION. Let n and k be non-negative integer numbers. The relation, given below involving three binomial coefficients (three elements of Pascal's triangle)

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

is called Stifel's relation.

Definition 2.5. PASCAL'S TRIANGLE. Let n and k be non-negative integer numbers. Pascal's triangle is an infinite numerical triangle formed by binomial coefficients, $\binom{n}{k}$, where n is associated with line number (vertical position), and k is associated with column number (horizontal position).

The Pascal's triangle can be seen as a triangular array of the binomial coefficients. Below, the first nine lines of Pascal's triangle (for more details, see [8]).

$n=0$	1								
$n=1$	1	1							
$n=2$	1	2	1						
$n=3$	1	3	3	1					
$n=4$	1	4	6	4	1				
$n=5$	1	5	10	10	5	1			
$n=6$	1	6	15	20	15	6	1		
$n=7$	1	7	21	35	35	21	7	1	
$n=8$	1	8	28	56	70	56	28	8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

3 Main result

We start this section presenting examples of two procedures that can be applied to the lines of Pascal's triangle resulting in numbers 'divisible by eleven. Then we formalize those procedures, state the main result as a theorem that applying any of those procedures to a line of Pascal's triangle result in a number divisible by eleven and prove it.

The idea for the main result of this manuscript first arose from the fact that the first lines (1 to 4, line 0 is excluded) of Pascal's triangle read directly as numbers provide values divisible by eleven, 11, 121, 1331 and 14641, all of then palindromes. On the other hand, that is not the case once you reach lines with elements greater than or equal to 10 (such as $\binom{5}{2}$). Motivated by this characteristic present in the first lines, we adapted the procedure to consider each element of the triangle as the coefficient of a decimal expansion instead of simply concatenating them. By doing so for a couple of lines further, regardless the direction of application, we are still generating values divisible by eleven.

For clarity, let us proceed to some examples.

Example 3.1. PASCAL'S TRIANGLE, $n = 8$.

$n=1$	1	1								11
$n=2$	1	2	1							11
$n=3$	1	3	3	1						11
$n=4$	1	4	6	4	1					11
$n=5$	1	5	10	10	5	1				
	1	5	10+1	0	5	1				
	1	6	1	0	5	1				11
$n=6$	1	6	15	20	15	6	1			
	1	6	10+5	10+10	10+5	6	1			
	1	6	15	20+1	5	6	1			
	1	6	15+2	1	5	6	1			
	1	7	7	1	5	6	1			11
$n=7$	1	7	21	35	35	21	7	1		
	1	7	21	35	37	1	7	1		
	1	7	21	38	7	1	7	1		
	1	7	24	8	7	1	7	1		
	1	9	4	8	7	1	7	1		11
$n=8$	1	8	28	56	70	56	28	8	1	
	2	1	4	3	5	8	8	8	1	11

Note that, starting from the left also results in a multiple of eleven. In short, it just reverses the order of the digits, still being divisible by eleven. In this case, $n=8$, the number 188853412 is also divisible by eleven.

Example 3.2. Pascal's triangle, $n=10$.

1	10	45	120	210	252	210	120	45	10	1
								45+1	0	1
							120+4	6	0	1
						210+12	4	6	0	1
					252+22	2	4	6	0	1
				210+27	4	2	4	6	0	1
			120+23	7	4	2	4	6	0	1
		45+14	3	7	4	2	4	6	0	1
	10+5	9	3	7	4	2	4	6	0	1
1+1	5	9	3	7	4	2	4	6	0	1
2	5	9	3	7	4	2	4	6	0	1

1	10	45	120	210	252	210	120	45	10	1
1	0	45+1								
1	0	6	120+4							
1	0	6	4	210+12						
1	0	6	4	2	252+22					
1	0	6	4	2	4	210+27				
1	0	6	4	2	4	7	120+23			
1	0	6	4	2	4	7	3	45+14		
1	0	6	4	2	4	7	3	9	10+5	
1	0	6	4	2	4	7	3	9	5	1+1
1	0	6	4	2	4	7	3	9	5	2

Both (25937424601 and 10642473952) are reverses of each other and divisible by eleven. A natural question that can be asked is whether this property is true for every line.

Before we present the procedures and prove the main result, we discuss explicitly the case $n=10$, as in diagrams in Example 3.2. We first introduce the notation: a_i

with $i = 0, 1, 2, \ldots$, is each digit appearing in the last line in EXAMPLE 3.2; V_i with $i = 0, 1, 2, \ldots$, is what is being mounted on the last line and c_i with $i = 0, 1, 2, \ldots$, is the value that is added to the next column of the diagram. Also, we definte $V_0 = 0 = c_0$. Therefore, we present only the four steps. Thus, for a_1, V_1 and c_1, we have

$$\begin{aligned} a_1 &= \binom{10}{10} + c_0 \pmod{10} \quad \longrightarrow \quad a_1 = 1, \\ V_1 &= 10^{1-1} \cdot a_1 + V_0 = 1 \\ c_1 &= \frac{1}{10}\left[\binom{10}{10-1+1} + c_0 - a_1\right] = 0. \end{aligned}$$

For a_2, V_2 and c_2, we obtain

$$\begin{aligned} a_2 &= \binom{10}{10-2+1} + c_1 \pmod{10} \quad \longrightarrow \quad a_2 = 0, \\ V_2 &= 10^{2-1} \cdot a_2 + V_1 = 1 \\ c_2 &= \frac{1}{10}\left[\binom{10}{10-2+1} + c_1 - a_2\right] = 1. \end{aligned}$$

Noting that, in EXAMPLE 3.2 this number $c_2 = 1$ is to be added to 45 on third column. For a_3, V_3 and c_3, we have

$$\begin{aligned} a_3 &= \binom{10}{10-3+1} + c_2 \pmod{10} \quad \longrightarrow \quad a_3 = 6, \\ V_3 &= 10^{3-1} \cdot a_3 + V_2 = 100 \cdot 6 + 1 = 601 \\ c_3 &= \frac{1}{10}\left[\binom{10}{10-3+1} + c_2 - a_3\right] = 4. \end{aligned}$$

Noting $V_3 = 601$ denotes the last line in EXAMPLE 3.2 being formed and $c_3 = 4$ the value to be added to 120 in the next (third column) column. We finally, consider a_4, V_4 and c_4. Thus, we have

$$\begin{aligned} a_4 &= \binom{10}{10-4+1} + c_3 \pmod{10} \quad \longrightarrow \quad a_4 = 4, \\ V_4 &= 10^{4-1} \cdot a_4 + V_3 = 4601 \\ c_4 &= \frac{1}{10}\left[\binom{10}{10-4+1} + c_3 - a_4\right] = 12. \end{aligned}$$

Also here, $V_4 = 4601$ denotes the last line in EXAMPLE 3.2 being formed and $c_4 = 12$ the value to be added to 210 in the next column (fourth column). Thus, for this case,

$n = 10$, we list the other ones (digits at last line) $V_5 = 24601$, $V_6 = 424601$, $V_7 = 7424601$, $V_8 = 37424601$, $V_9 = 937424601$, $V_{10} = 5937424601$ $V_{11} = 25937424601$. So, this las number is equal to 11^{10}, a well-known result, as we will see in Theorem 3.4

Now we understand the goal, let us formally define the procedures to how interpret elements of the Pascal's triangle line as a decimal expansion and the main result.

Procedure 3.1. Decimal expansion from right to left.

1. This procedure combines all the $n+1$ elements of n-th line of Pascal's triangle.
2. Define V_0 and c_0 to be 0.[ii]
3. For $k \geq 1$ we perform the following operations until for $k > n$ we have $a_k = 0$ and $c_k = 0$.
   (a) $a_k = \binom{n}{n-k+1} + c_{k-1} \pmod{10}$
   (b) $V_k = 10^{k-1} \cdot a_k + V_{k-1}$
   (c) $c_k = \frac{1}{10}\left(\binom{n}{n-k+1} + c_{k-1} - a_k\right)$
4. Note that c_i is a sequence of non-negative integers and for i greater than $n+1$ it is strictly decreasing. For this reason there is a j such that $\binom{n}{n-j+1} = 0$ and $c_{j-1} = 0$, which implies $a_j = 0$ and the procedure ends.

Procedure 3.2. Decimal expansion from left to right.

1. This procedure combines all the $n+1$ elements of n-th line of Pascal's triangle.
2. Define V_0 and c_0 to be 0.
3. For $k \geq 1$ we perform the following operations until for $k > n$ we have $a_k = 0$ and $c_k = 0$.
   (a) $a_k = \binom{n}{k-1} + c_{k-1} \pmod{10}$.
   (b) $V_k = 10 \cdot V_{k-1} + a_k$.
   (c) $c_k = \frac{1}{10}\left(\binom{n}{k-1} + c_{k-1} - a_k\right)$.
4. Note that c_i is a sequence non-negative integer and for i greater than $n+1$ its value is strictly decreasing. For this reason there is a j such that $\binom{n}{j-1} = 0$ and $c_{j-1} = 0$, which implies $a_j = 0$ and the procedure ends.

Theorem 3.3. *For any given line in the Pascal triangle Procedures 3.1 and 3.2 produce results that are reverses of each other.*

[ii] We used V because it is the result value and c because it works as a carrying value.

Proof 3.3.1. We will break the demonstration in three steps. First of all we will show that each iteration of *Procedure* 3.1 defines a digit of its final result, going from the least significant to the most. The second step is showing that the same stands for *Procedure* 3.2, but the digits are generated from the most significant to the least one. Finally we are going to prove that for a given k the values of a_k and c_k are the same for both procedures and once a_k is the next digit the procedures produce results that are the reverses of each other.

Part one: Each iteration of *Procedure* 3.1 defines one digit in the result. For this we will show by induction that $V_k < 10^k$ and that an iteration affects only the k-th digit of the result. As the base for the induction $V_0 = 0$ and $c_0 = 0$ imply that $a_1 = 1$, $V_1 = 1$ and $c_1 = 0$. So $V_1 < 10^1$ and the first step affected the first digit.

Now let's suppose that for a number k both assumptions hold. Then we have:

- $a_{k+1} = \binom{n}{n-k} + c_k \pmod{10} \implies 0 \leq a_{k+1} \leq 9$.
- $V_{k+1} = 10^k \cdot a_{k+1} + V_k < 9 \cdot 10^k + 10^k = 10^{k+1}$.
- $V_{k+1} = 10^k \cdot a_{k+1} + V_k$ and $V_k < 10^k$ implies that a_{k+1} is the $(k+1)^{\text{th}}$ digit from the least to the most significant in V_{k+1}.

So, by induction, the k-th digit of the result of *Procedure* 3.1 is a_k.

Part two: Each iteraction of *Procedure* 3.2 defines one digit in the result. This is simpler than the first step. As $0 \leq a_k \leq 9$ and $V_k = 10 \cdot V_{k-1} + a_k$ follows directly that the least significant digit of V_k is a_k, because V_{k-1} is multiplied by 10.

Part three: Results of *Procedures* 3.1 and 3.2 for the n-th line of Pascal triangle are reverses of each other. To prove this result we will show by induction that both procedures yield for a k-th step the same values of a_k and c_k. Doing so and considering steps 1 and 2 we will have shown the original proposition.

Let's use $V1$, $a1$ and $c1$ for the sequences generated by *Procedure* 3.1 and $V2$, $a2$ and $c2$ for those generated by *Procedure* 3.2.

Base of induction:

- $a1_1 = \binom{n}{n} + c_0 \pmod{10} \implies a1_1 = 1$
- $c1_1 = \frac{1}{10}\left(\binom{n}{n} + c_0 - a_1\right) = 0$
- $a2_1 = \binom{n}{0} + c_0 \pmod{10} \implies a2_1 = 1$
- $c2_1 = \frac{1}{10}\left(\binom{n}{0} + c_0 - a_1\right) = 0$

Let's suppose that for a given k we have $a1_k = a2_k$ and $c1_k = c2_k$. Thus follows that:

$$a2_{k+1} = \binom{n}{k} + c2_k \pmod{10} = \binom{n}{n-k} + c1_k \pmod{10} = a1_{k+1}$$

and

$$c2_{k+1} = \frac{1}{10}\left(\binom{n}{k} + c2_k - a2_k\right) = \frac{1}{10}\left(\binom{n}{n-k} + c1_k - a1_k\right) = c1_{k+1}$$

So by induction we have that sequences $c1$ and $c2$ are equal and sequences $a1$ and $a2$ as well, which concludes the demonstrantion that results yield by *Procedure* 3.1 and *Procedure* 3.2 are reverses. $\square$

Theorem 3.4. Any line of Pascal's triangle interpreted as a decimal expansion results in a multiple of eleven.

Proof 3.4.1. We shall separate the demonstration in two steps, one for each procedure.
(First step.) As described in *Procedure* 3.1, the result is the sum of the elements of the Pascal triangle line, in which the $n-k$ element is multiplied by the term 10^k:

$$\begin{array}{ll} \textit{First term} & \binom{n}{n} \cdot 10^{n-n}; \\ \\ \textit{Second term} & \binom{n}{n-1} \cdot 10^{n-(n-1)}; \\ \vdots & \qquad \vdots \\ (n+1)\text{-th } \textit{term} & \binom{n}{0} \cdot 10^{n-0}. \end{array}$$

Thus, the result R_n from *Procedure* 3.1 applied to line n-th is

$$R_n = \sum_{k=0}^{n} \binom{n}{k} \cdot 10^{n-k} \cdot$$

This expression is very similar to binomial theorem, by adding the power of 1 we achieve the exact format

$$R_n = \sum_{k=0}^{n} \binom{n}{k} \cdot 10^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 10^{n-k} \cdot 1^k = (10+1)^n = 11^n \cdot$$

Therefore, we conclude that the result of executing *Procedure* 3.1 for the n-th Pascal's triangle line is 11^n, and it follows naturally that R_n is divisible by eleven.

(Second step.) From the demonstration of **Theorem 3.3**, it is sufficient to show that the reverse of a multiple of eleven is also a multiple of eleven.

Let us consider a number N of k digits, each digit denoted by a_i with $0 \leqslant i \leqslant k-1, a_{k-1} \neq 0$. We also define S_o as the sum of the digits in odd positions and S_e as the sum of the digits in even position. Then we have

$$N = \sum_{i=0}^{k-1} a_i \cdot 10^i \equiv \sum_{i=0}^{k-1} a_i \cdot (-1)^i \equiv S_e - S_o \pmod{11}$$

So $11 \mid N \iff S_e \equiv S_o \pmod{11}$. Let N_r be the reverse of N and SR_o and SR_e be the sums of odd and even positioned digits, respectively, of N_r. Let us analyze the relations involving S_o, S_e, SR_o and SR_e.

The digit a_j from N is mapped to the digit a_{k-1-j} of N_r. The implications are considered in two cases.

Case 1. k is an even number, $(k = 2t)$. In this case we have

$$2 \mid k - 1 - j = 2t - 1 - j \implies 2 \mid j + 1 \implies 2 \nmid j.$$

Thus, every digit is mapped into the inverse parity, therefore $S_e = SR_o$ and $S_o = SR_e$. So,

$$S_e \equiv S_o \pmod{11} \iff SR_o \equiv SR_e \pmod{11} \implies 11 \mid N \iff 11 \mid N_r.$$

Case 2. k is an odd number, $(k = 2t + 1)$. Thus, in this case we have

$$2 \mid k - 1 - j = 2t + 1 - 1 - j \implies 2 \mid j.$$

So, every digit is mapped into the same parity, implying that $S_e = SR_e$ and $S_o = SR_o$. So,

$$S_e \equiv S_o \pmod{11} \iff SR_e \equiv SR_o \pmod{11} \implies 11 \mid N \iff 11 \mid N_r.$$

Therefore, for any N, we conclude that $11 \mid N_r \iff 11 \mid N$, i.e., if a number is divisible by eleven its reverse also.

Thus, we have that once the value generated by *Procedure* 3.1 is a multiple of eleven and the number generated by *Procedure* 3.2 is its reverse, we conclude that the value generated by *Procedure* 3.2 is also a multiple of eleven. □

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