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ARTICLE

Two deductions systems for the logic PM4N

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Abstract: The logic **PM4N** was introduced by Jean-Yves Beziau [1] as a modal and 4-valued system. In this introductory paper, the author presented the system from a matrix logic with four values disposed in a Boolean algebra with a modal operator for the notion of necessity. From that matrix semantics, the paper shows some valid results and stresses some motivations of the matrix semantics and of the system **PM4N**. In this paper we develop some additional aspects of that logic and present two deductive systems for it, a simple system of tableaux and a sequent calculus.

Keywords: Deduction systems; Many-valued logics; Tableaux; Sequent calculus.

Classification MSC: 03B53; 03B35

1 Introduction

Beziau [1] introduced the logic **PM4N** as a basic 4-valued and modal logic. The system was proposed to keep many important modal notions.

In this paper we propose some developments on that logic, considering details of its matrix semantics, preparing adequate deductive systems for **PM4N** and showing several modal laws valid in **PM4N**, but also laws and rules not valid in this 4-valued system.

The layout of our paper is as follows. In Section 2, we remember the 4-valued semantic of **PM4N**, some motivations and mathematical aspects of that model. In Section 3, we elaborate a tableaux system for **PM4N** and show the validity of many modal laws. In Section 4, we introduce a sequent calculus for **PM4N**. Finally, some conclusions from this work are drawn in Section 5.

2 The four-valued semantic

The logic **PM4N** is built over a propositional language determined by the set of operators $L = \{\vee, \neg, \Box\}$, in which the propositional operators $\vee$, $\neg$ and $\Box$ denote, respectively, the notions of disjunction, negation and necessity.

We will use the same symbols from the propositional language over the semantic structure for **PM4N**.

In the introductory paper [1], the author considered as basic also the operator $\wedge$, but since $\wedge$ and $\vee$ are inter-definable in **PM4N**, we have considered only the three operators above.

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Like [1], the original semantic for **PM4N** is defined by the following matrix semantics:

$$\mathcal{M}_{\mathbf{PM4N}} = (\{0, n, b, 1\}, \vee, \neg, \Box, \{b, 1\}),$$

such that b and 1 are the designated values, and 0 and n are the non-designated values.

Remark 2.1. The set of designated or true values is denoted by $D = \{b, 1\}$.

Before presenting the tables of these operators let us show the algebraic motivation, since we think that with these elements the next developments will follow clearly.

We must consider these four values disposed in a Boolean algebra of four elements as this one:

$$\begin{array}{ccccc} & & 1 & & \\ & / & & \backslash & \\ n & & & & b \\ & \backslash & & / & \\ & & 0 & & \end{array}$$

Now it is good to think about the operators. The disjunction corresponds to the supremum, that is, for $x, y \in \{0, n, b, 1\}$, the element $x \vee y = sup\{x, y\}$. The negation corresponds to the boolean complement, and the necessitation that only the value 1 is necessarily true.

So, the meaning of the basic operators are in Figure 1.

Fig. 1. Tables for the operators $\vee$, $\neg$ and $\Box$

$\vee$	0	n	b	1
0	0	n	b	1
n	n	n	1	1
b	b	1	b	1
1	1	1	1	1

	$\neg$
0	1
n	b
b	n
1	0

	$\Box$
0	0
n	0
b	0
1	1

Source: Adapted from [1]

Beziau [1] used the symbols $\{0^-, 0^+, 1^-, 1^+\}$ for $\{0, n, b, 1\}$, and some papers have used the set $\{f, n, b, t\}$. Our choice is also frequent in the literature about logics with four values.

We have a four-valued logic and as usual the values 0 and 1 represent the *falsun* and the *verun*, but b and n are two complementary values that are not in a linear order.

Some papers, as in Font [2] and Omori and Sano [3], use the intuition that n means neither 0 and nor 1, and b means both 0 and 1. Beziau considered 0 as necessarily false, n as possibly false, b as possibly true, and 1 as necessarily true.

This negation is a type of classical negation, for it maps true values into false values and maps false values into true values.

Following some motivations in algebraic developments on Belnap's four-valued logics [4] and in the literature about algebraic and mathematical logics [5], [6], [7] and [8], we present additional algebraic aspects of **PM4N**.

Of course, the algebraic structure of **PM4N** is isomorphic with the Boolean algebra of four elements, $\mathcal{P}(2) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, generated by a set of two elements $2 = \{0, 1\}$.

Another way to see this partial order is to consider the isomorphic structure in $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, considering that in $\mathbb{Z}_2$, $1 = [1]$ and $0 = [0]$, odd and even, respectively.

$$\begin{array}{ccccc} & & 1 = (1,1) & & \\ & \diagup & & \diagdown & \\ n = (0,1) & & & & b = (1,0) \\ & \diagdown & & \diagup & \\ & & 0 = (0,0) & & \end{array}$$

With the following operations:

$(x, y) \wedge (z, w) = (x.z, y.w)$

$\sim (x, y) = (1 - x, 1 - y)$ and

$(x, y) \vee (z, w) = \sim [\sim (x, y) \wedge \sim (z, w)]$.

Thus, we can present the formal matrix semantics of **PM4N**.

We write $Var(\textbf{PM4N}) = \{p_1, p_2, p_3, ...\}$ for the set of the propositional variables of **PM4N** and For(**PM4N**) for the set of the formulas of **PM4N**, defined in the usual way.

Definition 2.2. A restrict valuation for **PM4N** is a function

$$v : Var(\textbf{PM4N}) \to \{0, n, b, 1\}.$$

Definition 2.3. A valuation for **PM4N** is a function that extends, in an unique way, the restrict valuation for the whole set For(**PM4N**), according to the matrix above.

Definition 2.4. A formula $\varphi \in For(\textbf{PM4N})$ is valid, according to $\mathcal{M}_{PM4N}$, if, for every **PM4N**-valuation v, $v(\varphi) \in D$.

Definition 2.5. If $\Gamma \cup \{\varphi\} \subseteq For(\textbf{PM4N})$, then Γ implies logically φ, or φ is a semantic consequence of Γ, when, for every **PM4N**-valuation v, if $v(\Gamma) \subseteq D$, then $v(\varphi) \in D$.

Thus, we have that for every valuation v:

$$\Gamma \vDash \varphi \iff (v(\Gamma) \subseteq D \Rightarrow v(\varphi) \in D).$$

Based on the literature of many-valued logics [9], [10] and besides these primitive and basic operators, we can define the following operators in **PM4N**:

Possibility: $\Diamond x =_{def} \neg\Box\neg x$

Conjunction: $x \wedge y =_{def} \neg(\neg x \vee \neg y)$

Conditional: $x \to y =_{def} \neg x \vee y$

Biconditional: $x \leftrightarrow y =_{def} (x \to y) \wedge (y \to x)$

Consistency: $\circ x =_{def} \Box x \vee \neg\Diamond x$

Inconsistency (or Contingency): $\bullet x =_{def} \Diamond x \wedge \neg\Box x$

Fig. 2. Tables for the new operators defined from the primitive and basics operators

	$\Diamond$
0	0
n	1
b	1
1	1

$\wedge$	0	n	b	1
0	0	0	0	0
n	0	n	0	n
b	0	0	b	b
1	0	n	b	1

$\rightarrow$	0	n	b	1
0	1	1	1	1
n	b	1	b	1
b	n	n	1	1
1	0	n	b	1

$\leftrightarrow$	0	n	b	1
0	1	b	n	0
n	b	1	0	n
b	n	0	1	b
1	0	n	b	1

	$\circ$
0	1
n	0
b	0
1	1

	$\bullet$
0	0
n	1
b	1
1	0

	$\sim$
0	1
n	n
b	b
1	0

Source: By the authors

Paraconsistent negation: $\sim x =_{def} \neg x \leftrightarrow \circ x$

The meanings of these new operators are given by the Figure 2.

By the tables for $\neg$, $\wedge$, $\vee$ and the definition of disjunction, the De Morgan's laws hold:

$$\neg(x \vee y) = (\neg x \wedge \neg y) \text{ and } \neg(x \wedge y) = (\neg x \vee \neg y).$$

The operator of possibility is defined as usual in the context of modal logics. It is a type of dual of the necessity, and the interpretation is that if a proposition is not necessarily false, then it is possible.

The conjunction is defined by the De Morgan laws and if taken the conjunction as the basic operator, then we could define the disjunction from it. The interpretation is given by the infimum between the two values.

The conditional operator or implication is defined as in the classical case by the negation of antecedent or the consequent; and the biconditional is a double conditional.

In modal logics, a possible but not necessary proposition is called contingent and in the logic **PM4N**, its meaning coincides with the meaning of inconsistency of logics of formal inconsistency (LFI) as we can see in Carnielli, Coniglio and Marcos [11], Carnielle and Marcos [12] and Carnielli, Marcos and Amo [13]. The operators of consistency and inconsistency are complementary and an interpretation for the consistency is that the classical values $\{0, 1\}$ are consistent and the non-classical $\{n, b\}$ are not consistent.

Finally, the negation $\sim$ is paraconsistent [14] because it takes the true value b into b, and the false value n into n. So in some cases a proposition and its negation can be true.

Given the tables for $\Box$, $\Diamond$ and $\neg$ we have the following equivalences:

$$\Box\neg x = \neg\Diamond x \text{ and } \Diamond\neg x = \neg\Box x.$$

We can see these modal operators in a hexagon of oppositions as the following figure:

$\circ x$

$$
\begin{array}{ccc}
\Box x & & \Box\neg x = \neg\Diamond x \\
| & & | \\
| & & | \\
\Diamond x & & \Diamond\neg x = \neg\Box x \\
& \diagdown \quad \diagup & \\
& \bullet x &
\end{array}
$$

We can see the oppositions in the diagonals between necessary and non-necessary, possible and impossible, and contingent and non-contingent, but we do not have a global order of deduction in the figure. In the vertical lines, the deduction goes top-down and in other lines the deduction goes down-top.

Another interesting operation that can be defined in **PM4N** is the following addition:

Ring addition: $x \oplus y =_{def} (x \wedge \neg y) \vee (\neg x \wedge y)$, with the table:

Fig. 3. Table for the operation $\oplus$

$\oplus$	0	n	b	1
0	0	n	b	1
n	n	0	1	b
b	b	1	0	n
1	1	b	n	0

Source: By the authors

We can observe that 0 is the neutral element for $\oplus$, each element is its own symmetric, and the operation is commutative.

Finally, we can show that $(\{0, n, b, 1\}, \oplus, \wedge, 0, 1)$ is a commutative ring with identity element 1.

As an observation, $(\{0, n, b, 1\}, \oplus, 0, 1)$ is the Klein's group of four elements.

3 A tableaux system for PM4N

From the matrix model for **PM4N** we elaborated a completely adequate tableaux system for this logic [15] and [16].

We denote this system by $\mathcal{T}_{\mathbf{PM4N}}$.

We have tried tableaux with fewer than four ramifications as usual for four-valued logics. In many cases we succeeded, but not always.

As a first step we will take the following notations: $f \in \{0, n\}$ for false values, and $t \in \{b, 1\}$ for true values.

We will use signed formulas as $k \quad \psi$ to indicate that the formula ψ has a value $k \in \{0, n, b, 1\}$. And we will denote the Boolean complement of $k \in \{0, n, b, 1\}$ by k'.

$f - t$ expansions:

$$
\frac{f \quad \varphi}{0 \ \varphi \quad | \quad n \ \varphi} \qquad\qquad \frac{t \quad \varphi}{b \ \varphi \quad | \quad 1 \ \varphi}
$$

Negation:

$$\begin{array}{ll} k & \neg\varphi \\ \hline k' & \varphi \end{array}$$

Conjunction:

$$\begin{array}{lclclcl} 0 & & & \varphi\wedge\psi & & & \\ \hline 0\ \ \varphi & | & 0\ \ \psi & | & b\ \ \varphi & | & n\ \ \varphi \\ & | & & | & n\ \ \psi & | & b\ \ \psi \end{array} \qquad \begin{array}{lclcl} n & & \varphi\wedge\psi & & \\ \hline & \multicolumn{2}{c}{n\ \ \varphi} & | & 1\ \ \varphi \\ n\ \ \psi & | & 1\ \ \psi & | & n\ \ \psi \end{array}$$

$$\begin{array}{lcl} b & \varphi\wedge\psi & \\ \hline b\ \ \varphi & | & 1\ \ \varphi \\ t\ \ \psi & | & b\ \ \psi \end{array} \qquad \begin{array}{ll} 1 & \varphi\wedge\psi \\ \hline 1 & \varphi \\ 1 & \psi \end{array}$$

Disjunction:

$$\begin{array}{ll} 0 & \varphi\vee\psi \\ \hline 0 & \varphi \\ 0 & \psi \end{array} \qquad \begin{array}{lcl} n & \varphi\vee\psi & \\ \hline n\ \ \varphi & | & 0\ \ \varphi \\ f\ \ \psi & | & n\ \ \psi \end{array} \qquad \begin{array}{lclcl} b & & \varphi\vee\psi & & \\ \hline & \multicolumn{2}{c}{b\ \ \varphi} & | & 0\ \ \varphi \\ b\ \ \psi & | & 0\ \ \psi & | & b\ \ \psi \end{array}$$

$$\begin{array}{lclclcl} 1 & & & \varphi\vee\psi & & & \\ \hline 1\ \ \varphi & | & 1\ \ \psi & | & b\ \ \varphi & | & n\ \ \varphi \\ & | & & | & n\ \ \psi & | & b\ \ \psi \end{array}$$

Conditional:

$$\begin{array}{ll} 0 & \varphi\to\psi \\ \hline 1 & \varphi \\ 0 & \psi \end{array} \qquad \begin{array}{lcl} n & \varphi\to\psi & \\ \hline b\ \ \varphi & | & 1\ \ \varphi \\ f\ \ \psi & | & n\ \ \psi \end{array} \qquad \begin{array}{lclcl} b & & \varphi\to\psi & & \\ \hline & \multicolumn{2}{c}{n\ \ \varphi} & | & 1\ \ \varphi \\ 0\ \ \psi & | & b\ \ \psi & | & b\ \ \psi \end{array}$$

$$\begin{array}{lclclcl} 1 & & & \varphi\to\psi & & & \\ \hline 0\ \ \varphi & | & 1\ \ \psi & | & n\ \ \varphi & | & b\ \ \varphi \\ & | & & | & n\ \ \psi & | & b\ \ \psi \end{array}$$

And we are particularly interested in the modal operators, for the author of [1] at introducing **PM4N** desired to compare it with the modal logic **S5**.

So using centrally the following rules we can show some laws of **S5** [17] that are valid for **PM4N**, but also several results of rupture.

Modal operators:

$$\begin{array}{ll} 1 & \Box\varphi \\ \hline 1 & \varphi \end{array} \qquad \begin{array}{ll} 0 & \Diamond\varphi \\ \hline 0 & \varphi \end{array} \qquad \begin{array}{lcl} 0 & \Box\varphi & \\ \hline f\ \ \varphi & | & b\ \ \varphi \end{array} \qquad \begin{array}{lcl} 1 & \Diamond\varphi & \\ \hline n\ \ \varphi & | & t\ \ \varphi \end{array}$$

Definition 3.1. A branch of a tableau of $\mathcal{T}_{\mathbf{PM4N}}$ is closed if the marked formulas occur in the path:

(i) $k_1\ \ \varphi$ and $k_2\ \ \varphi$, for any formula φ and $k_1 \neq k_2$;

(ii) $n\ \ \Box\varphi$ or $b\ \ \Box\varphi$, for any formula $\Box\varphi$;

(iii) $n\ \ \Diamond\varphi$ or $b\ \ \Diamond\varphi$, for any formula $\Diamond\varphi$.

The $f - t$ expansions were introduced to reduce the length of the trees. From the item (i) above, branches with $f\ \ \psi$ and $t\ \ \psi$ are closed.

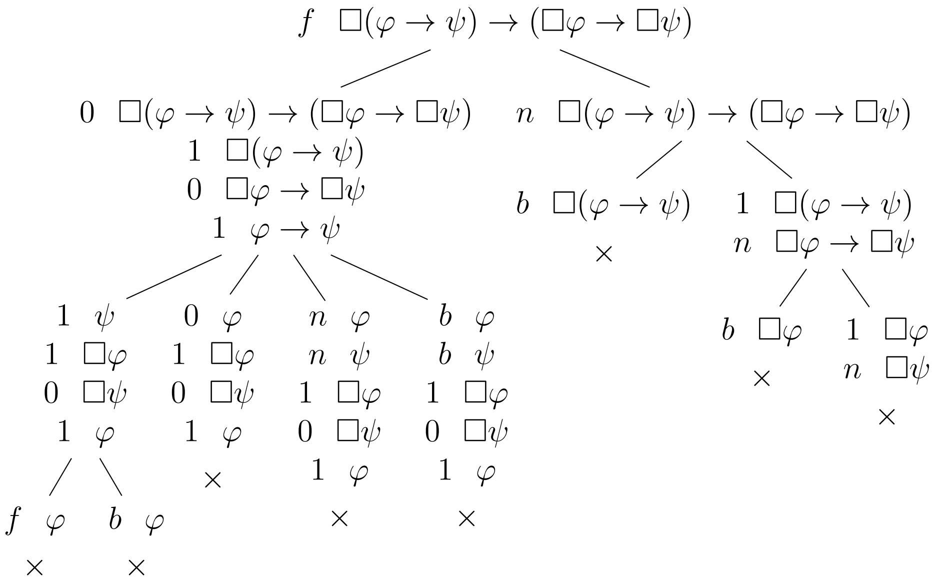
Definition 3.2. A tableau of $\mathcal{T}_{\mathbf{PM4N}}$ is closed if all of its branches are closed.

The proofs and deductions in $\mathcal{T}_{\mathbf{PM4N}}$ are as the usual. We denote that a formula φ is deduced from Γ in the tableau system $\mathcal{T}_{\mathbf{PM4N}}$ by $\Gamma \Vdash \varphi$.

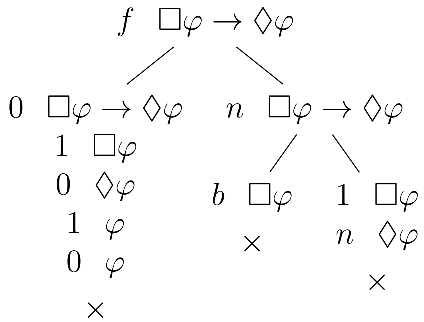
The deduction $\Gamma \Vdash \varphi$ is Tarskian, that is, (i) if $\varphi \in \Gamma$, then $\Gamma \Vdash \varphi$; (ii) if $\Gamma \Vdash \varphi$ and $\Gamma \subseteq \Sigma$, then $\Sigma \Vdash \varphi$; (iii) if $\Gamma \Vdash \varphi$ and $\Sigma \cup \{\varphi\} \Vdash \psi$, then $\Gamma \cup \Sigma \Vdash \psi$.

Now, considering the tableaux system above we can show the validity of several modal sentences, but also the non-validity of other central modal results.

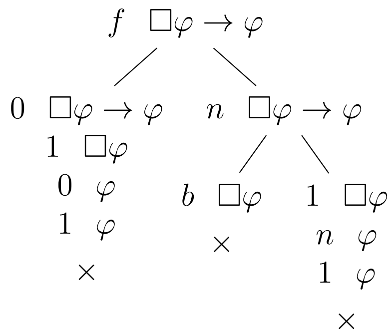
(a) **K**: $\Vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$



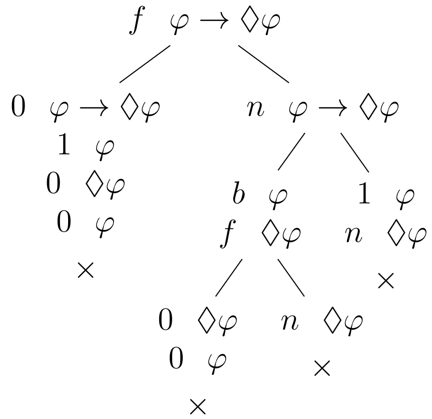
(b) **D**: $\Vdash \Box\varphi \to \Diamond\varphi$



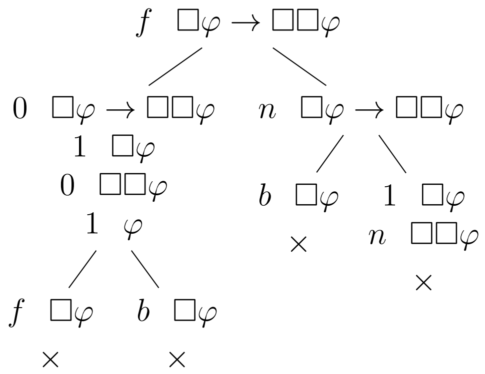
(c) **T**: $\Vdash \Box\varphi \to \varphi$



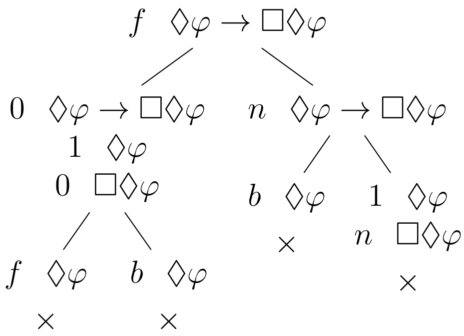
(d) **T'**: $\Vdash \varphi \to \Diamond\varphi$



(e) **4**: $\Vdash \Box\varphi \to \Box\Box\varphi$



(f) **4'**: $\Vdash \Diamond\Diamond\varphi \to \Diamond\varphi$
(g) **5**: $\Vdash \Diamond\varphi \to \Box\Diamond\varphi$



(h) **5'**: $\Vdash \Diamond\Box\varphi \to \Box\varphi$

As consequence of the previous laws, we have:

(i) **B**: $\Vdash \varphi \to \Box\Diamond\varphi$

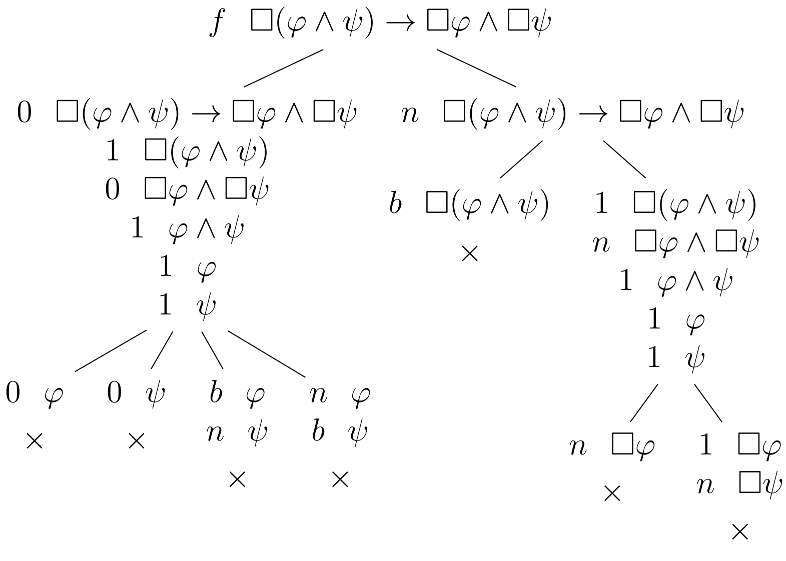
(j) **B'**: $\Vdash \Diamond\Box\varphi \to \varphi$

The distributivity of modal operators to the conjunction and disjunction would be another interesting property to verify.

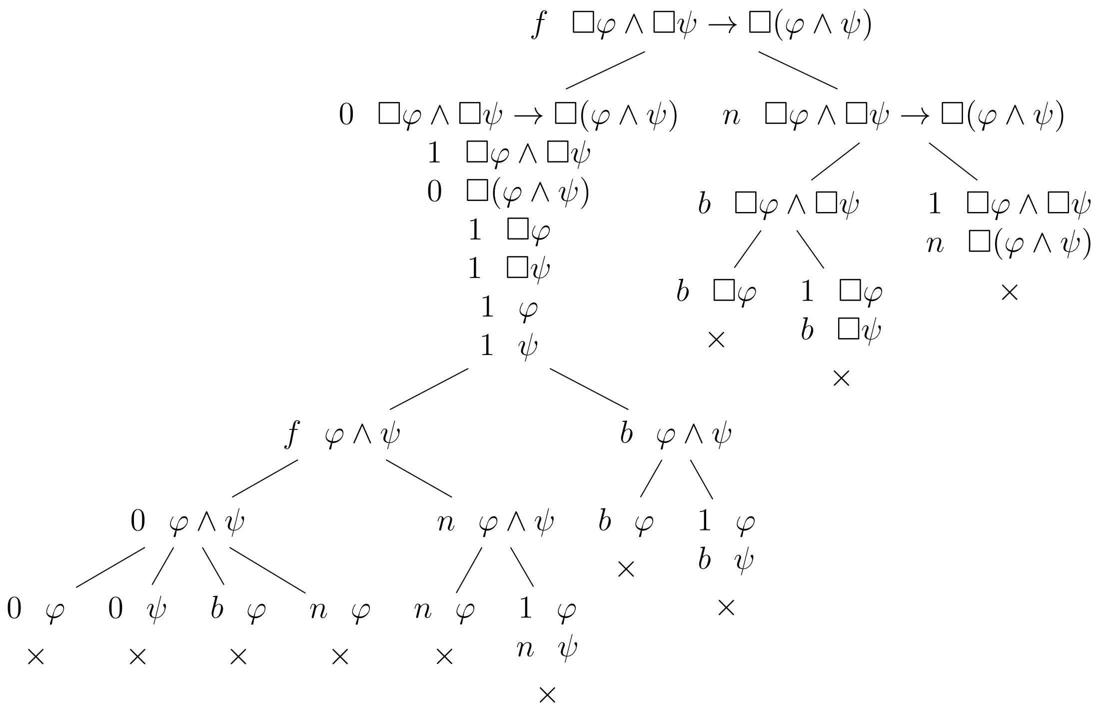
From the definition of the biconditional we can divide the formula below into two sub-formulas and ascertain their validity.

(k) **R**: $\Vdash \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$

M: $\Vdash \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$



C: $\Vdash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$



(l) **R'**: $\Vdash \Diamond\varphi \vee \Diamond\psi \leftrightarrow \Diamond(\varphi \vee \psi)$

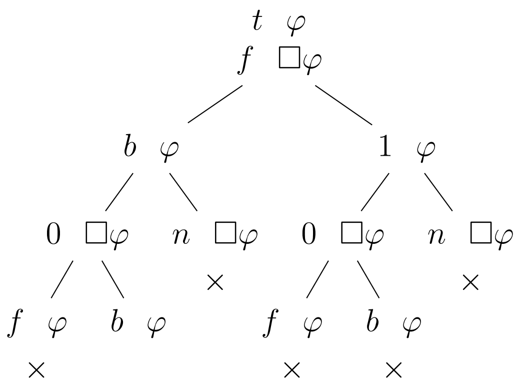
(m) $\nVdash \Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$

(n) $\nVdash \Diamond\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi)$

For (m) and (n) we can just take the values $v(\varphi) = b$ and $v(\psi) = n$.

However, as its mentioned, the two systems must be different. We begin with the non-validity of the Necessitation Rule.

(o) **RN**: $\varphi \nVdash \Box\varphi$, even when φ is valid in **PM4N**.



There is an open branch, exactly when $v(\varphi) = b$. So the tableau is not closed only when $v(\varphi) = b$, and if the formula φ takes only the value 1, then it is necessary and the rule can be applicable.

(p) **RM**: $\varphi \to \psi \nVdash \Box\varphi \to \Box\psi$

For $v(\varphi) = 1$ and $v(\psi) = b$, we have $1 \to b$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Box\varphi \to \Box\psi) = 0$.

(q) **RM'**: $\varphi \to \psi \nVdash \Diamond\varphi \to \Diamond\psi$

For $v(\varphi) = n$ and $v(\psi) = 0$, we have $n \to 0$ and $1 \to 0$, then $v(\varphi \to \psi) = b$ and $v(\Diamond\varphi \to \Diamond\psi) = 0$.

The non-validity of these rules makes the conception of this logic a little estrange. A valid conditional does not transfer its validity to these several cases.

In a further paper we must to compare the set of theorems of **PM4N** and **S5**, the modal logic of equivalence relation.

But, for this moment we are just interested in the deductions systems for **PM4N**. Thus, in the next section we construct a sequent calculus for this logic.

4 From matrixes to sequents

In this section we will introduce a sequent system for **PM4N**, following principally [18] and [19].

We start considering the matrix semantics of **PM4N**:

$$\mathcal{M}_{\mathbf{PM4N}} = (\{0, n, b, 1\}, \vee, \neg, \Box, \{b, 1\}),$$

with $D = \{b, 1\}$.

Definition 4.1. A sequent for **PM4N** is a figure Λ of the type:

$$\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1,$$

such that for $x \in \{0, n, b, 1\}$, each Λ_x is a finite sequence of formulas. Each Λ_x is a component of Λ.

Some of these components can be the empty sequence $\emptyset$.

If Λ and Λ' are sequences, then Λ, Λ' is their union respecting each component.

The sequence $\Lambda_0 \mid \Lambda_n$ is the antecedent and the sequence $\Lambda_b \mid \Lambda_1$ is the succedent of Λ.

As usual, we think of the elements of the antecedent as a conjunction and the elements of the succedent as a disjunction.

As the figure suggests, a sequent must hold if some element of the antecedent is false, or if some element of the succedent is true.

Definition 4.2. A valuation v satisfies a sequent Λ or is a model of Λ if there is $x \in \{0, n, b, 1\}$ and $\varphi \in \Lambda_x$ such that $v(\varphi) = x$.

We denote that v is a model of Λ by $v \vDash \Lambda$.

Sometimes we can abuse this notion and say that the valuation v satisfies the component Λ_x, when $\varphi \in \Lambda_x$ and $v(\varphi) = x$.

Definition 4.3. The sequent Λ is satisfiable if there is a valuation v so that $v \vDash \Lambda$.

Definition 4.4. The sequent Λ is valid if every valuation v satisfies Λ.

For $\Phi \cup \{\varphi\} \subseteq For(\mathbf{PM4N})$ and a valuation $v : For(\mathbf{PM4N}) \rightarrow \{0, n, b, 1\}$ we consider: $\Phi_0 = \{\psi \in \Phi : v(\psi) = 0\}$, $\Phi_n = \{\psi \in \Phi : v(\psi) = n\}$, $\{\varphi\}_b = \{\varphi\}$, if $v(\varphi) = b$ or $\{\varphi\}_b = \emptyset$, if $v(\varphi) \neq b$, and $\{\varphi\}_1 = \{\varphi\}$, if $v(\varphi) = 1$ or $\{\varphi\}_1 = \emptyset$, if $v(\varphi) \neq 1$.

In this case, if $v(\varphi) = 1$, then the sequent $\Phi_0 \mid \Phi_n \Rightarrow \emptyset \mid \{\varphi\}$ is satisfiable.

Considering that a sequent is composed by a finite number of formulas, then the set Φ in the next proposition must be finite too.

Proposition 4.5. *$\Phi \vDash \varphi$ iff the sequent $\Phi_0 \mid \Phi_n \Rightarrow \{\varphi\}_b \mid \{\varphi\}_1$ is valid.*
Proof: *If $\Phi \vDash \varphi$, then every model of Φ is also a model of φ. Thus, for a generic valuation v such that $v(\Phi) \subseteq D$, we have that $v(\varphi) = b$ or $v(\varphi) = 1$. In one way, the sequent $\Phi_0 \mid \Phi_n \Rightarrow \{\varphi\}_b \mid \{\varphi\}_1$ is valid.*

If $\Phi \nvDash \varphi$, then there exists a valuation v such that $v(\Phi) \subseteq D$ and $v(\varphi) = n$ or $v(\varphi) = 0$. For this valuation v we have that $\Phi_0 \mid \Phi_n \Rightarrow \{\varphi\}_b \mid \{\varphi\}_1$ is not valid. ■

As $\{\neg\neg\varphi\} \vDash \varphi$, we have that $\{\neg\neg\varphi\} \mid \{\neg\neg\varphi\} \Rightarrow \{\varphi\} \mid \{\varphi\}$ is valid, for any valuation v, if $v(\varphi) = x$, then v satisfies Λ_x.

As $\{\Box\varphi\} \vDash \varphi$, we have that $\{\Box\varphi\} \mid \{\Box\varphi\} \Rightarrow \{\varphi\} \mid \{\varphi\}$ is valid, because for any valuation v, if $v(\varphi) = 1$, then v satisfies Λ_1 and if $v(\varphi) \neq 1$, then v satisfies Λ_0.

The sequent $\{\varphi\} \mid \{\varphi\} \Rightarrow \{\Box\varphi\} \mid \{\Box\varphi\}$ is not valid because if $v(\varphi) = b$, then none of the components is valid.

From the Gentzen tradition and according to Zach [19], we must present valid axioms and structural rules that preserve validity.

To be valid in this systems it is enough a valuation v such that there is a $\varphi \in \Lambda_x$ with $v(\varphi) = x$.

So the axioms are of the form $\{\varphi\} \mid \{\varphi\} \Rightarrow \{\varphi\} \mid \{\varphi\}$.

For any valuation v, one of the components of Λ is valid.

In the following we will not use the curly brackets anymore.

The rules must be such that if the premisses (upper sequents) are valid, then its consequence (lower sequent) is valid too.

For the structural rules, we will present the cases in the second component, but the rules are correct in any component.

Weakening: $\dfrac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}$

Contraction: $\dfrac{\Lambda_0 \mid \Lambda_n, \varphi, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}$

Exchange: $\dfrac{\Lambda_0 \mid \Lambda_n, \varphi, \psi \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \psi, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}$

Cut: $\dfrac{\Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1 \quad \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$

In all of these rules if the premisses are valid, then their consequences are also valid.

The cut is a binary rule with two premisses and, if both are valid, the valuation that operates does not depend on φ and the lower sequent is valid too.

Now we will present the sequent rules for the basic operators $\neg, \Box$ and $\vee$ and justify that these rules preserve validity. These rules introduce operators in the sequents.

Sequent rules:

Negation

$$\text{(R1)} \quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi}{\Lambda_0, \neg\varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1} \qquad \text{(R2)} \quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \neg\varphi \Rightarrow \Lambda_b \mid \Lambda_1}$$

$$\text{(R3)} \quad \frac{\Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \neg\varphi \mid \Lambda_1} \qquad \text{(R4)} \quad \frac{\Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \neg\varphi}$$

For any valuation v, if the premisses are valid, then the consequences are also valid, just observing that if $v(\varphi) = x$, then $v(\neg\varphi) = x'$.

All of them are unary rules, with exactly one upper sequent. For the other operators will occur rules with more than one upper sequent, and in these cases we consider exactly one of them as valid.

Necessity

$$\text{(R5)} \quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \Box\varphi}$$

$$\text{(R6)} \quad \frac{\Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1; \;\; \Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1; \;\; \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0, \Box\varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$$

For any valuation v, if $v(\varphi) = 1$, then $v(\Box\varphi) = 1$, and if $v(\varphi) \neq 1$, then $v(\Box\varphi) = 0$. The two rules preserve validity.

The next rules, for the disjunction, can have up to four sequents as premisses.

Disjunction

$$\text{(R7)} \quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi; \;\; \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \psi; \;\; \Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b, \psi \mid \Lambda_1; \;\; \Lambda_0 \mid \Lambda_n, \psi \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi \vee \psi}$$

$$\text{(R8)} \quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi, \psi \mid \Lambda_1; \;\; \Lambda_0, \psi \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1; \;\; \Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b, \psi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \vee \psi \mid \Lambda_1}$$

$$\text{(R9)} \quad \frac{\Lambda_0 \mid \Lambda_n, \varphi, \psi \Rightarrow \Lambda_b \mid \Lambda_1; \;\; \Lambda_0, \varphi \mid \Lambda_n, \psi \Rightarrow \Lambda_b \mid \Lambda_1; \;\; \Lambda_0, \psi \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \varphi \vee \psi \Rightarrow \Lambda_b \mid \Lambda_1}$$

$$\text{(R10)} \quad \frac{\Lambda_0, \varphi, \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0, \varphi \vee \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$$

If the premisses for some of these rules are valid, then for any valuation v exactly one component of upper sequents must be valid. In any case, using the table for disjunction the consequence must be valid too.

So the axioms of this sequent system are valid and the rules preserve the validity.

Definition 4.6. The proof of a sequent $\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1$ in this calculus is a tree of sequents such that every topmost sequent is an axiom, all other sequents in the tree are obtained from the sequents immediately above it using one of the rules, and the sequent $\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1$ is the end-sequent, at the root of the tree.

Definition 4.7. A sequent Λ is provable if it is the end-sequent of some proof.

Examples:

(a) $\neg\neg\varphi \mid \neg\neg\varphi \Rightarrow \varphi \mid \varphi$

$$\dfrac{\dfrac{\overline{\varphi \mid \varphi \Rightarrow \varphi \mid \varphi} \quad (\text{Axiom})}{\emptyset \mid \emptyset \Rightarrow \neg\varphi, \varphi \mid \neg\varphi, \varphi} \quad (\text{R3}) \text{ and } (\text{R4})}{\neg\neg\varphi \mid \neg\neg\varphi \Rightarrow \varphi \mid \varphi} \quad (\text{R1}) \text{ and } (\text{R2})$$

(b) $\Box\varphi \mid \Box\varphi \Rightarrow \varphi \mid \varphi$

$$\dfrac{\dfrac{\overline{\varphi \mid \varphi \Rightarrow \varphi \mid \varphi} \quad (\text{Axiom})}{\Box\varphi \mid \emptyset \Rightarrow \varphi \mid \varphi} \quad (\text{R6})}{\Box\varphi \mid \Box\varphi \Rightarrow \varphi \mid \varphi} \quad (\text{Weakening})$$

(c) $\Box\varphi \vee \Box\psi \mid \emptyset \Rightarrow \emptyset \mid \Box(\varphi \vee \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi \mid \varphi \Rightarrow \varphi \mid \varphi} \quad (\text{Axiom})}{\Box\varphi \mid \emptyset \Rightarrow \emptyset \mid \varphi} \quad (\text{R6})}{\Box\varphi \mid \emptyset \Rightarrow \emptyset \mid \varphi \vee \psi} \quad (\text{R7}) \qquad \dfrac{\dfrac{\overline{\psi \mid \psi \Rightarrow \psi \mid \psi} \quad (\text{Axiom})}{\Box\psi \mid \emptyset \Rightarrow \emptyset \mid \psi} \quad (\text{R6})}{\Box\psi \mid \emptyset \Rightarrow \emptyset \mid \varphi \vee \psi} \quad (\text{R7})}{\Box\varphi \vee \Box\psi \mid \emptyset \Rightarrow \emptyset \mid \varphi \vee \psi} \quad (\text{R10})}{\Box\varphi \vee \Box\psi \mid \emptyset \Rightarrow \emptyset \mid \Box(\varphi \vee \psi)} \quad (\text{R5})$$

Theorem 4.8. *(Soundness) If a sequent of **PM4N** is provable, then it is valid.*
Proof: *By induction on the length of proofs. As observed, the axioms are valid sequents, and the rules preserve the validity. Then each provable sequent is valid.* ■

Completeness also can be met in Zach [19]. The idea is due to Schutte (*apud* [19]), that for a sequent Λ either it has a proof or it has a counter-model.

Now it is important to observe that each sequent is a 4-sequence of finite sequences. So they are always finite.

Theorem 4.9. *(Completeness) If a sequent of **PM4N** is valid, then it is provable.*
Proof: *We begin with a sequent Λ. From it we must construct a tree of sequents $\boldsymbol{T}$ that can represent a proof of Λ or show a counter-model for Λ.*

We test Λ. If it is an axiom, then it has a proof. If not, we begin the construction of the tree down-up or the reduction tree in an induction way.

First step: Write Λ at the root of the tree.

Induction step: If the topmost sequent Δ of a branch includes only atomic formulas p, then stop the construction for this branch.

If the sequent Δ does not have only atomic formulas, then it is an open branch. Thus Δ includes some formula with a principal operator. So we must consider the rule that originated this operator in the tree.

We repeat the following steps for each non-atomic formula ψ that appears at the i_{th} step in the topmost sequent Δ of an open branch, if it was not reduced at step i on this branch, nor it is the result of a reduction at this stage.

This formula ψ is a negation or a necessitation or a disjunction, due to its principal operator.

This way the reduction in this action seems to:

$$\frac{\Delta', \sigma}{\Delta}$$

such that at the formula(s) σ is applied the rule of introduction for the principal operator of ψ. The formula ψ occurs in Δ and Δ' is the part standard for the two sequents.

When this process is concluded, let $\boldsymbol{T}$ be the reduction tree edified.

As Λ is composed by a finite number of formulas and the reductions go over the sub-formulas of formulas in previous step, then the concluded tree $\boldsymbol{T}$ is finite.

This way, each leaf sequent includes only atomic formulas that occur at every place in that sequent.

If in each leaf is an axiom, like we have used only introduction rules for $\boldsymbol{T}$, then we have a proof that is a cut-free proof from axioms using the formulas detached in $\boldsymbol{T}$ and Λ has a proof without the cut.

If the sequent in some leaf is not an axiom, then we define a valuation for each of these variables putting $v(p) = y \neq x$, for Λ_x. By induction on the complexity of any formula $\psi \in \boldsymbol{B}$, no formula ψ in $\boldsymbol{B}$ takes the truth-value corresponding to the position for which it stands. Hence v does not satisfy Λ and v is a counter-model for the original sequent Λ. ■

In the spirit of these developments, we can present more results for the other operators too.

Sequent rules:

Possibility

$$\text{(R11)}\quad \frac{\Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \Diamond\varphi}$$

$$\text{(R12)}\quad \frac{\Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0, \Diamond\varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$$

Conjunction

$$\text{(R13)}\quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi, \psi}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi \wedge \psi}$$

$$\text{(R14)}\quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi, \psi \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1, \psi;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \psi \mid \Lambda_1, \varphi}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \wedge \psi \mid \Lambda_1}$$

$$\text{(R15)}\quad \frac{\Lambda_0 \mid \Lambda_n, \varphi, \psi \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1, \psi;\ \ \Lambda_0 \mid \Lambda_n, \psi \Rightarrow \Lambda_b \mid \Lambda_1, \psi}{\Lambda_0 \mid \Lambda_n, \varphi \wedge \psi \Rightarrow \Lambda_b \mid \Lambda_1}$$

$$\text{(R16)}\quad \frac{\Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0, \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b, \psi \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n, \psi \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0, \varphi \wedge \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$$

Conditional

$$\text{(R17)}\quad \frac{\Lambda_0, \varphi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \psi;\ \ \Lambda_0 \mid \Lambda_n, \varphi, \psi \Rightarrow \Lambda_b \mid \Lambda_1;\ \ \Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi, \psi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi \to \psi}$$

$$\text{(R18)}\quad \frac{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \psi \mid \Lambda_1, \varphi;\ \ \Lambda_0 \mid \Lambda_n, \varphi \Rightarrow \Lambda_b, \psi \mid \Lambda_1;\ \ \Lambda_0, \psi \mid \Lambda_n, \varphi \Rightarrow \Lambda_b \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \to \psi \mid \Lambda_1}$$

$$\text{(R19)}\quad \frac{\Lambda_0 \mid \Lambda_n, \psi \Rightarrow \Lambda_b \mid \Lambda_1, \varphi;\ \ \Lambda_0 \mid \Lambda_n, \psi \Rightarrow \Lambda_b, \varphi \mid \Lambda_1;\ \ \Lambda_0, \psi \mid \Lambda_n \Rightarrow \Lambda_b, \varphi \mid \Lambda_1}{\Lambda_0 \mid \Lambda_n, \varphi \to \psi \Rightarrow \Lambda_b \mid \Lambda_1}$$

$$\text{(R20)}\quad \frac{\Lambda_0, \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1, \varphi}{\Lambda_0, \varphi \to \psi \mid \Lambda_n \Rightarrow \Lambda_b \mid \Lambda_1}$$

Definition 4.10. A proof of a sentence φ is a proof of a sequent as $\emptyset \mid \emptyset \Rightarrow \{\varphi\}_b \mid \{\varphi\}_1$.

Now the proof for the scheme **T**: $\Box\varphi \to \varphi$.

(i) $\emptyset \mid \emptyset \Rightarrow \Box\varphi \to \varphi \mid \Box\varphi \to \varphi$

$$\cfrac{\cfrac{\cfrac{\overline{\varphi \mid \varphi \Rightarrow \varphi \mid \varphi} \quad \text{(Axiom)}}{\Box\varphi \mid \emptyset \Rightarrow \varphi \mid \varphi \quad \text{(R6)}}}{\Box\varphi \mid \Box\varphi \Rightarrow \varphi \mid \varphi \quad \text{(Weakening)}}}{\emptyset \mid \emptyset \Rightarrow \Box\varphi \to \varphi \mid \Box\varphi \to \varphi \quad \text{(R17) and (R18)}}$$

5 Final considerations

The aim of this paper was to understand and contribute with the development of the modal and 4-valued logic **PM4N**.

We reviewed the matrix semantics of **PM4N** and made explicit its algebraic character. We presented in the usual way a tableaux system for this logic. Finally, we introduced a sequent calculus for that logic and showed the adequacy of the sequent calculus for the matrix semantic of **PM4N**. In a further paper we must compare the system **PM4N** with other well known logical systems.

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