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On compact explicit formulas of the partial fraction decomposition and applications

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Abstract: This study concerns another approach for computing the scalars $A_i^{(k)}$ of the partial fraction decomposition $F(x) = \dfrac{R(x)}{Q(x)} = \dfrac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i}\frac{A_i^{(k)}}{(x-\gamma_i)^k}$, where $R(x)$ and $Q(x)$ are polynomials of real or complex coefficients, with $\deg(R) < \deg(Q)$. More precisely, we provide a method to exhibit compact explicit formulas of the scalars $A_i^{(k)}$ $(1 \leq i \leq s,\ 1 \leq k \leq m_i)$. Some illustrative special cases and several examples are furnished, to show the efficiency of this new approach. Finally, concluding remarks and perspectives are presented.

Keywords: Partial fraction decomposition, Derivation of higher order, Explicit formula.

Classification MSC: Primary 26A33; 30C15 Secondary 26C99

1 Introduction

Since a long time, the partial fraction decomposition (or expansion) represents a fundamental topic, with several applications in various fields of pure and applied mathematics such that differential equations, control theory, and other fields of applied sciences and engineering. In addition, the partial fraction decomposition is also an important tool in the teaching areas such that Calculus, Laplace transform, etc. It is worth noting that the literature is very vast on this very rich topic.

A well known theoretical theorem in algebra asserts that every rational function has a unique partial fraction decomposition (see, for example, [3, 4, 10, 11]). Moreover, several methods have been improved in the literature for computing the partial fraction decomposition. Generally, there are two usual approaches for establishing the numerators of the partial fractions, namely, for computing the scalars $A_i^{(k)}$. The technique of the first approach, known as the "*Method of undetermined coefficients*", consists first in reducing

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to the same denominator the partial fraction decomposition, of the given rational function. Second, since the denominators on both sides are the same, the numerators must also be the same. Then, we equalize the similar coefficients (corresponding to the same power of x) of the two polynomials of the numerators on either side of the equality[i]. Therefore, the scalars $A_i^{(k)}$ can be found by solving a system of linear equations. The second approach is based on the application of the Heaviside's "*cover-up method*", which necessitate substitutions to establish the scalars $A_i^{(k)}$, of the partial fraction decomposition, in the case with single poles γ_j $(1 \leq j \leq \deg(Q))$. For multiple poles case γ_j $(1 \leq j \leq s)$, with $m_j \geq 2$ for some j, successive differentiation are applied, for calculating the scalars $A_i^{(k)}$. Despite that, this topic continue to attract much attention, and there has been recent developments in the computation aspect of the scalars $A_i^{(k)}$, for general rational functions (see for example, [3, 4, 6, 15]) as well as for some special cases (see, for example, [5, 6, 12–14]). Meanwhile, the approaches and methods for decomposing a rational function into partial fractions are computationally intensive, especially when the multiplicities of roots of the denominator are higher.

In this paper we establish another approach for providing the explicit formulas for the scalars $A_i^{(k)}$ of the partial fraction decomposition of the rational functions $F(x) = \frac{R(x)}{Q(x)}$, where $R(x)$, $Q(x)$ are polynomials in $\mathbb{R}[X]$ or $\mathbb{C}[X]$, such that (without loss of generality) the degree of R is less than the degree of Q and are mutually prime. The essence of our approach requires a computational process, based on two known results of the literature. More precisely, suppose that R and Q are mutually prime and $Q(x) = \prod_{j=1}^{s}(x-\gamma_j)^{m_j}$, where each root γ_j is of multiplicity $m_j \geq 1$. We develop a computational process, which allows us to present a new method, for exhibiting compact explicit formulas of the partial fraction decomposition,

$$F(x) = \frac{R(x)}{Q(x)} = \frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \frac{A_i^{(k)}}{(x-\gamma_i)^k}.$$

Our main goal is to give a new compact explicit formulas for the scalars $A_i^{(k)}$ $(1 \leq i \leq s$, $1 \leq k \leq m_j)$. As a consequence, some applications and several illustrative examples are presented, in order to show the efficiency of our approach.

This study is organized as follows. For reason of clarity and conciseness, Section 2 is devoted to the two fundamental results, representing the basic tools of our method. Section 3 is concerned with the generic case, where a compact explicit formula of the

[i] It is known that two polynomials are equal, if and only if, the coefficients at the corresponding powers of x are equal.

partial fraction decomposition is given for the special case $F(x) = \frac{1}{Q(x)}$. In Section 4 we study the general case of partial fraction decomposition, where the explicit formulas of the partial fraction decomposition of the generic case plays a central key. In addition, two special cases are provided. Results of Sections 3 and 4 are illustrated by significant examples, in order to show the efficiency of our approach. Finally, in Section 5 we give some concluding remarks and perspective.

For reason of clarity, in the sequel we suppose that for the rational fraction $F(x) = \frac{R(x)}{Q(x)}$, the two polynomials are relatively prime, namely, the great common divisor of $R(x)$ and $q(x)$ is equal to 1.

2 Two fundamental results

In this section we consider the rational fraction $F(x) = \frac{R(x)}{Q(x)}$, where $Q(x) = x^r + a_0x^{r-1} + \cdots + a_{r-1}$, and without loss of generality we suppose $\deg(R) < \deg(Q)$. Let γ_i $(1 \leq i \leq s)$ be the distinct roots (real or complex) of the polynomial $Q(x)$, with multiplicities m_i $(1 \leq i \leq s)$, respectively. Then, we have $Q(x) = \prod_{i=1}^{s}(x - \gamma_i)^{m_i}$.

In sequel of our study, the next known result, will play a central role for computing the scalars $A_i^{(k)}$.

Theorem 2.1. *Let $F(x) = \frac{R(x)}{Q(x)}$ be a rational function such that $Q(x) = \prod_{i=1}^{s}(x - \gamma_i)^{m_i}$ and deg$(R) < \deg(Q) = m_1 + ... + m_s$, where the γ_i $(1 \leq i \leq s)$, with $\gamma_i \neq \gamma_j$ for $i \neq j$, are real or complex numbers. Then, the partial fractional decomposition of $F(x)$, is given by,*

$$F(x) = \frac{R(x)}{\prod_{i=1}^{s}(x - \gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \frac{A_i^{(k)}}{(x - \gamma_i)^k},$$

where

$$A_i^{(k)} = \frac{1}{(m_i - k)!}\left[\frac{R(x)}{\prod_{j=1,j\neq i}^{s}(x - \gamma_j)^{m_j}}\right]_{x=\gamma_i}^{(m_i-k)}.$$

Here $[f(x)]_{x=\gamma}^{(n)} = \frac{d^n f(x)}{dx^n}|_{x=\gamma} = f^{(n)}(\gamma)$, which means the value of the derivation of order n of the function f at $x = \gamma$.

Result of Theorem 2.1 is well known in the literature and it has been demonstrated by various algebraic and analytical methods (see, for example, [3, 4, 15]).

The second classical result, which will allow us to reach our goal, concerns a generalization, relating to the derivation of order $d \geq 2$ of a product of differentiable functions.

Theorem 2.2. *Let f_1, f_2,..., f_s be derivable functions, until the order d. Then, we have,*

$$\left[\prod_{j=1}^{s} f_j\right]^{(d)} = \sum_{h_1+\ldots+h_s=d} \binom{d}{h_1 \ldots h_s} \prod_{j=1}^{s} f_j^{(h_j)}, \tag{1}$$

where $f^{(k)}$ means the derivation de order k of the function f and $\binom{d}{h_1 \ldots h_s} = \frac{d!}{h_1! \ldots h_s!}$.

Theorem 2.2 is not common in books of Calculus or real analysis. In fact, it represents a generalization of the well- known formula of the Calculus $(fg)' = f'g + fg'$. A known generalization, of this former expression, has been established for $s = 2$ and $d \geq 2$, and it is given by,

$$(fg)^{(d)} = \sum_{h_1+h_2=d} \binom{d}{h_1\ h_2} f^{(h_1)} g^{(h_2)} = \sum_{h=0}^{d} \binom{d}{h} f^{(h)} g^{(d-h)}. \tag{2}$$

The former Formula Eq. (2) can be established by induction, analogously to that which makes it possible to establish the Newton's binomial formula, namely, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. Similarly, Formula Eq. (1) can be also established by induction, by adapting the proof of the classical formula

$$[a_1 + a_2 + \ldots a_s]^{(d)} = \sum_{h_1+\ldots+h_s=d} \binom{d}{h_1 \ldots h_s} \prod_{j=1}^{s} a_j^{(h_j)}.$$

Theorem 2.2 and Formula Eq. (2), have been studied in [1, 8]. They are considered as generalization of Leibniz's rule for differentiation.

3 Partial fraction decomposition: Generic case and special cases

A Study of the generic case In this section we consider a monic (or unitary) polynomial $Q(x) = x^r + a_0 x^{r-1} + \cdots + a_{r-1}$. Let γ_i $(1 \leq i \leq s)$ be the distinct roots of $Q(x)$, with multiplicities m_i $(1 \leq i \leq s)$, respectively. Then, we have $Q(x) = \prod_{i=1}^{s} (x-\gamma_i)^{m_i}$. For the partial fractional decomposition of $F(x) = \frac{1}{Q(x)}$, a direct application of Theorem 2.1 (with $R(x) = 1$) shows easily that we have,

$$F(x) = \frac{1}{\prod_{i=1}^{s} (x-\gamma_i)^{m_i}} = \sum_{i=1}^{s} \sum_{k=1}^{m_i} \frac{A_i^{(k)}}{(x-\gamma_i)^k}. \tag{3}$$

with

$$A_i^{(k)} = \frac{1}{(m_i-k)!}\left[\frac{1}{\prod_{j=1,j\neq i}^{s}(x-\gamma_j)^{m_j}}\right]_{x=\gamma_i}^{(m_i-k)}. \tag{4}$$

In the aim to obtain the compact explicit expression of the scalars $A_i^{(k)}$, let apply Theorem 2.2 by taking $f_j(x) = \frac{1}{(x-\gamma_j)^{m_j}} = (x-\gamma_j)^{-m_j}$. More precisely, we apply the preceding Formula Eq. (4) to the family of functions $H_i(x)$ $(1 \leq i \leq s)$ defined by,

$$H_i(x) = \prod_{j=1,j\neq i}^{s} f_j(x) = \prod_{j=1,j\neq i}^{s}(x-\gamma_j)^{-m_j}, \text{ for } 1 \leq i \leq s.$$

For reason of clarity and simplicity, $\Gamma_{i,d_i(k)} = \{[h_j]_i = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_s);\ \sum_{1\leq j\neq i\leq s} h_j = d_i(k)\}$ and we set $d_i(k) = m_i - k$. Then, the main formula of Theorem 2.2, allows us to derive that,

$$[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = \sum_{[h_j]_i\in\Gamma_{i,k}} \binom{d_i(k)}{h_1..., h_{i-1}, h_{i+1}, ...h_s} \prod_{j=1,j\neq i}^{s} f_j^{(h_j)}(\gamma_i), \tag{5}$$

where $\binom{d_i(k)}{h_1...\hat{h}_i...h_s} = \frac{d_i(k)!}{h_1!...h_{i-1}!h_{i+1}!...h_s!}$. It is well known that the derivative $f_j^{(h_j)}(x) = [(x-\gamma_j)^{-m_j}]^{(h_j)}$ is given explicitly by $f_j^{(h_j)}(x) = (-1)^{h_j}\frac{(h_j+m_j-1)!}{(m_j-1)!}(x-\gamma_j)^{-m_j-h_j}$. Thus, we derive that $f_j^{(h_j)}(\gamma_i) = (-1)^{h_j}\frac{(h_j+m_j-1)!}{(m_j-1)!}(\gamma_i-\gamma_j)^{-m_j-h_j}$. By substitution of this former formula of $f_j^{(h_j)}(x)$ in Expression Eq. (5) of $[H_i(x)]_{x=\gamma_i}^{(d_i(k))}$, a straightforward computation allows us to deduce that

$$[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = \sum_{[h_j]_i\in\Gamma_{i,k}} \binom{d_i(k)}{h_1..., h_{i-1}, h_{i+1}, ...h_s} \prod_{j=1,j\neq i}^{s} f_j^{(h_j)}(\gamma_i).$$

Therefore, we get

$$[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = \sum_{[h_j]_i\in\Gamma_{i,k}} \binom{d_i(k)}{h_1..., h_{i-1}, h_{i+1}, ...h_s} \prod_{j=1,j\neq i}^{s} (-1)^{h_j}\frac{(h_j+m_j-1)!}{(m_j-1)!}(\gamma_i-\gamma_j)^{-m_j-h_j}.$$

And a direct computation shows that, we have,

$$[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = \sum_{\Gamma_{i,d_i(k)}} (-1)^{m_i-k}(m_i-k)! \prod_{j=1,j\neq i}^{s} \binom{h_j+m_j-1}{h_j}(\gamma_i-\gamma_j)^{-m_j-h_j},$$

or equivalently,

$$[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = (-1)^{m_i-k}(m_i-k)! \sum_{[h_j]_i \in \Gamma_{i,d_i(k)}} \Omega_i([h_j]_i;[\gamma_j]), \tag{6}$$

where $[h_j]_i = (h_1,...,h_{i-1},h_{i+1},...,h_s)$, $[\gamma_j] = (\gamma_1,...,\gamma_s)$ and

$$\Omega_i([h_j]_i;[\gamma_j]) = \prod_{j=1,j\neq i}^{s} \left[\binom{h_j+m_j-1}{h_j} (\gamma_i-\gamma_j)^{-m_j-h_j} \right]. \tag{7}$$

Consequently, first, we observe that Expressions Eq. (3)-Eq. (4), Theorem 2.1 and Theorem 2.2, imply that the partial fraction decomposition of the function $F(x) = Q(x)^{-1}$ is given by,

$$\frac{1}{Q(x)} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \frac{A_i^{(k)}}{(x-\gamma_i)^k} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \frac{1}{(m_i-k)!}\frac{[H_i(x)]_{x=\gamma_i}^{(d_i(k))}}{(x-\gamma_i)^k}.$$

Second, taking into account Expressions Eq. (6)-Eq. (7), we get $A_i^{(k)} = \frac{1}{(m_i-k)!}[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = \frac{1}{(m_i-k)!} \times (-1)^{m_i-k}(m_i-k)! \sum_{[h_j]_i\in\Gamma_{i,d_i(k)}} \Omega_i([h_j]_i;[\gamma_j])$, which implies that we have,

$$A_i^{(k)} = \frac{1}{(m_i-k)!}[H_i(x)]_{x=\gamma_i}^{(d_i(k))} = (-1)^{m_i-k} \sum_{[h_j]_i \in \Gamma_{i,d_i(k)}} \Omega_i([h_j]_i;[\gamma_j]),$$

where the scalars $\Omega_i([h_j]_i;[\gamma_j])$ are as in Expression Eq. (7). In summary, the previous discussion shows that the main result of this section, can be formulated as follows.

Theorem 3.1. *Under the preceding data the partial fraction decomposition of the rational function* $F(x) = \frac{1}{Q(x)}$, *where* $Q(x) = \prod_{i=1}^{s}(x-\gamma_i)^{m_i}$, *is given by,*

$$F(x) = \frac{1}{Q(x)} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \left[(-1)^{m_i-k} \sum_{[h_j]_i \in \Gamma_{i,d_i(k)}} \Omega_i([h_j]_i;[\gamma_j]) \right] \frac{1}{(x-\gamma_i)^k}, \tag{8}$$

where $\Gamma_{i,d_i(k)} = \{[h_j]_i = (h_1,...,h_{i-1},h_{i+1},...,h_s);\ \sum_{1\leq j\neq i\leq s} h_j = m_i-k\}$ *and the* $\Omega_i([h_j]_i;[\gamma_j])$ *are yielded by Expression* Eq. (7), *namely, we have* $F(x) = \frac{1}{Q(x)} = \sum_{i=1}^{s}\sum_{k=1}^{m_i} \frac{A_i^{(k)}}{(x-\gamma_i)^k}$, *where*

$$A_i^{(k)} = (-1)^{m_i-k} \sum_{[h_j]_i \in \Gamma_{i,d_i(k)}} \Omega_i([h_j]_i;[\gamma_j]). \tag{9}$$

In Section 4, Expressions Eq. (8)-Eq. (9) of Theorem 3.1, will be utilized for studying the general case of the partial fraction decomposition of a rational function $F(x) = \frac{R(x)}{Q(x)}$, where $Q(x) = \prod_{i=1}^{s}(x-\gamma_i)^{m_i}$ and $R(x)$ is a polynomial, such that (without loss of generality) $\deg(R) < \deg(Q) = m_1 + m_2 + \ldots + m_s$. That is, the results of this subsection combined with Theorem 2.2, will be extensively used for providing explicit compact formulas, for the scalars $A_i^{(k)}$, of the partial fraction decomposition in the general setting.

Finally, it is important to note that a formula analogous to Eq. (8), has been highlighted in [2], using another approach.

B Special cases and illustrative examples This subsection is devoted to illustrate the efficiency of the result of the main Theorem 3.1. We study with more details the special case $s = 2$, along with various numerical examples. To this aim, we start by the following proposition.

Proposition 3.2. Consider the polynomial $Q(x) = (x-\gamma_1)^{m_1}(x-\gamma_2)^{m_2}$, where γ_1, γ_2 are real or complex numbers and m_1, m_2 are positive integer. Then, the partial fraction decomposition of the rational function $F(x) = \frac{1}{Q(x)}$, is given by,

$$F(x) = \frac{1}{Q(x)} = \sum_{k=1}^{m_1} \frac{A_1^{(k)}}{(x-\gamma_1)^k} + \sum_{k=1}^{m_2} \frac{A_2^{(k)}}{(x-\gamma_2)^k}, \tag{10}$$

where

$$\begin{cases} A_1^{(k)} = (-1)^{m_1-k}\Omega_{2,k}(\gamma_1,\gamma_2) = (-1)^{m_1-k}\binom{m_2+m_1-k-1}{m_2-k}(\gamma_2-\gamma_1)^{-(m_1+m_2)+k}, \\ A_2^{(k)} = (-1)^{m_2-k}\Omega_{2,k}(\gamma_1,\gamma_2) = (-1)^{m_2-k}\binom{m_1+m_2-k-1}{m_2-k}(\gamma_2-\gamma_1)^{-(m_1+m_2)+k}. \end{cases} \tag{11}$$

Proof. We apply result of Theorem 3.2. We have $A_1^{(k)} = (-1)^{m_1-k}\sum_{\Gamma_{1,k}}\Omega_{1,k}(\gamma_1,\gamma_2)$, where $\Gamma_{1,k} = \{(h_2);\ h_2 = m_1-k\} = \{m_1-k\}$, which implies that, we have $\Omega_{1,k}(\gamma_1,\gamma_2) = \binom{h_2+m_2-1}{h_2}(\gamma_1-\gamma_2)^{-h_2-m_2}$. Therefore, we obtain,

$$\Omega_{1,k}(\gamma_1,\gamma_2) = \binom{m_1+m_2-(k+1)}{m_1-k}(\gamma_1-\gamma_2)^{-(m_1+m_2)+k}.$$

Similarly, since $A_2^{(k)} = (-1)^{m_2-k}\sum_{\Gamma_{2,k}}\Omega_{2,k}(\gamma_1,\gamma_2)$ and $\Gamma_{2,k} = \{(h_1);\ h_1 = m_2-k\} = \{m_2-k\}$, we derive $\Omega_{2,k}(\gamma_1,\gamma_2) = \binom{h_1+m_1-1}{h_1}(\gamma_2-\gamma_1)^{-h_1-m_1}$, which implies that we have,

$$\Omega_{2,k}(\gamma_1,\gamma_2) = \binom{m_1+m_2-(k+1)}{m_2-k}(\gamma_2-\gamma_1)^{-(m_1+m_2)+k}.$$

Therefore, we obtain the partial fraction decomposition Eq. (10), with $A_1^{(k)}$ and $A_2^{(k)}$ given by Eq. (11). □

Suppose that $m_1 = m_2 = 1$ and consider the function $F(x) = \frac{1}{(x-\gamma_1)(x-\gamma_2)}$, where γ_1 and γ_2 are in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$. Then, its partial fraction decomposition is given by $F(x) = \frac{A_1^{(1)}}{x-\gamma_1} + \frac{A_2^{(1)}}{x-\gamma_2}$. Let apply our process for computing the scalars $A_1^{(1)}$ and $A_2^{(1)}$. Since $m_1 = 1$ and $m_2 = 1$, we can show that $\Gamma_{1,1} = \{1-1\} = \{0\}$ and $\Gamma_{2,1} = \{1-1\} = \{0\}$. Hence, we get $\Omega_{1,1}(\gamma_1,\gamma_2) = \binom{1+1-1-1}{1-1}(\gamma_1 - \gamma_2)^{-(1+1)+1} = \frac{1}{\gamma_1-\gamma_2}$ and $\Omega_{1,2}(\gamma_1,\gamma_2) = \binom{1+1-1-1}{1-1}(\gamma_2-\gamma_1)^{-(1+1)+1} = \frac{1}{\gamma_2-\gamma_1}$. Therefore, we have $A_1^{(1)} = (-1)^{1-1}\sum_{\Gamma_{1,1}}\Omega_{1,1}(\gamma_1,\gamma_2) = \frac{1}{\gamma_1-\gamma_2}$ and $A_1^{(2)} = (-1)^{1-1}\sum_{\Gamma_{1,2}}\Omega_{1,1}(\gamma_1,\gamma_2) = \frac{1}{\gamma_2-\gamma_1}$. Accordingly, we state the following corollary.

Corollary 3.3. Let γ_1 and γ_2 be in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$. Then, the partial fraction decomposition of the rational functions $F(x) = \frac{1}{(x-\gamma_1)(x-\gamma_2)}$, is given by,

$$F(x) = \frac{1}{(x-\gamma_1)(x-\gamma_2)} = \frac{1}{\gamma_1-\gamma_2}\frac{1}{x-\gamma_1} + \frac{1}{\gamma_2-\gamma_1}\frac{1}{x-\gamma_2}. \tag{12}$$

Let consider the following illustrative numerical example of Expression Eq. (12).

Example 3.4. Let $Q(x) = (x-8)(x-6)$, and consider the rational fration $F(x) = \frac{1}{(x-8)(x-6)}$. Here we have $\gamma_1 = 8$, $\gamma_2 = 6$. And by applying Expression Eq. (12), we show that the partial fraction decomposition of the rational function $F(x) = \frac{1}{(x-8)(x-6)}$, is given by, $F(x) = \frac{1}{2}.\frac{1}{x-8} - \frac{1}{2}.\frac{1}{x-6}$.

This decomposition is consistent with the direct calculation, which shows the efficiency of Theorem 3.1 and its Corollary 3.3.

Suppose $m_1 = 2$ and $m_2 = 1$ and consider the rational function $F(x) = \frac{1}{(x-\gamma_1)^2(x-\gamma_2)}$, where γ_1, γ_2 in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$. Then, its partial fraction decomposition can be written under the form $F(x) = \frac{A_1^{(1)}}{x-\gamma_1} + \frac{A_1^{(2)}}{(x-\gamma_1)^2} + \frac{A_2^{(1)}}{x-\gamma_2}$. Let apply our process for computing the scalars $A_1^{(1)}$, $A_1^{(2)}$ and $A_2^{(1)}$. Since $m_1 = 2$ and $m_2 = 1$, we can show that $\Gamma_{1,1} = \{1-1\} = \{0\}$, $\Gamma_{1,2} = \{1-1\} = \{0\}$ and $\Gamma_{2,1} = \{1-1\} = \{0\}$. A direct application of Expression Eq. (11), allows us to show that $\Omega_{1,1}(\gamma_1,\gamma_2) = \binom{2+1-(1+1)}{2-1}(\gamma_1 - \gamma_2)^{-(2+1)+1} = \frac{1}{(\gamma_1-\gamma_2)^2}$, $\Omega_{1,2}(\gamma_1,\gamma_2) = \binom{2+1-(2+1)}{2-2}(\gamma_1-\gamma_2)^{-(2+1)+2} = \frac{1}{\gamma_1-\gamma_2}$

and $\Omega_{2,1}(\gamma_2,\gamma_1) = \binom{2+1-(1+1)}{1-1}(\gamma_2-\gamma_1)^{-(1+2)+1} = \frac{1}{(\gamma_2-\gamma_1)^2}$. Therefore, have $A_1^{(1)} = (-1)^{2-1}\sum_{\Gamma_{1,1}}\Omega_{1,1}(\gamma_1,\gamma_2) = -\frac{1}{(\gamma_1-\gamma_2)^2}$, $A_1^{(2)} = (-1)^{2-2}\sum_{\Gamma_{1,2}}\Omega_{1,2}(\gamma_1,\gamma_2) = \frac{1}{\gamma_1-\gamma_2}$ and $A_2^{(1)} = (-1)^{1-1}\sum_{\Gamma_{2,1}}\Omega_{2,1}(\gamma_1,\gamma_2) = \frac{1}{(\gamma_2-\gamma_1)^2}$. Finally, we can state the following corollary.

Corollary 3.5. Let γ_1 and γ_2 be in $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$. Then, the partial fraction decomposition of the rational functions $F(x) = \frac{1}{(x-\gamma_1)^2(x-\gamma_2)}$, is given by,

$$F(x) = -\frac{1}{(\gamma_1-\gamma_2)^2}.\frac{1}{x-\gamma_1} + \frac{1}{\gamma_1-\gamma_2}.\frac{1}{(x-\gamma_1)^2} + \frac{1}{(\gamma_2-\gamma_1)^2}.\frac{1}{x-\gamma_2}. \qquad (13)$$

Let consider the following illustrative numerical example of Expression Eq. (13).

Example 3.6. Consider $F(x) = \frac{1}{(x-8)^2(x-6)}$. Since $\gamma_1 = 8$, $\gamma_2 = 6$, $m_1 = 2$ and $m_2 = 1$, a direct computation implies that we have $A_1^{(1)} = -\sum_{\Gamma_{1,1}}\Omega_{1,1}(8,6) = -\frac{1}{4}$, $A_1^{(2)} = \sum_{\Gamma_{1,1}}\Omega_{1,2}(8,6) = \frac{1}{2}$ and $A_2^{(1)} = \sum_{\Gamma_{2,1}}\Omega_{2,1}(6,8) = -\frac{1}{2}$. Therefore, Expression Eq. (13) implies that the partial fraction decomposition of the rational function $F(x) = \frac{1}{(x-8)^2(x-6)}$, is given by, $F(x) = -\frac{1}{4}.\frac{1}{x-8} + \frac{1}{2}.\frac{1}{(x-8)^2} + \frac{1}{4}.\frac{1}{x-6}$.

This partial fraction decomposition is consistent with the direct calculation, using other known methods, which shows the efficiency of Theorem 3.1 and its Corollary 3.5.

Suppose that $m_1 = 2$ and $m_2 = 3$ and consider the function $F(x) = \frac{1}{(x-\gamma_1)^2(x-\gamma_2)^3}$, where γ_1 and γ_2 are in $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$, then, its partial fraction decomposition is given by $F(x) = \frac{A_1^{(1)}}{x-8} + \frac{A_1^{(2)}}{(x-8)^2} + \frac{A_2^{(1)}}{x-6} + \frac{A_2^{(2)}}{(x-6)^2} + \frac{A_2^{(3)}}{(x-6)^3}$. Let apply our process for computing the scalars $A_1^{(1)}$, $A_1^{(2)}$, $A_2^{(1)}$, $A_2^{(2)}$ and $A_2^{(2)}$. Since $m_1 = 2$ and $m_2 = 3$, we can show that $\Gamma_{1,1} = \{2-1\} = \{1\}$, $\Gamma_{1,2} = \{2-2\} = \{0\}$, $\Gamma_{2,1} = \{3-1\} = \{2\}$, $\Gamma_{2,2} = \{3-2\} = \{1\}$ and $\Gamma_{2,3} = \{3-3\} = \{0\}$. An analogous straightforward computation allows us to establish that,

$$\begin{cases} \Omega_{1,1}(\gamma_1,\gamma_2) = \binom{2+3-(1+1)}{2-1}(\gamma_1-\gamma_2)^{-(2+3)+1} = \frac{3}{(\gamma_1-\gamma_2)^4}, \\ \Omega_{1,2}(\gamma_1,\gamma_2) = \binom{2+3-(2+1)}{2-2}(\gamma_1-\gamma_2)^{-(2+3)+2} = \frac{1}{(\gamma_1-\gamma_2)^3}, \\ \Omega_{2,1}(\gamma_2,\gamma_1) = \binom{2+3-(1+1)}{3-1}(\gamma_2-\gamma_1)^{-(2+3)+1} = 3.(\gamma_2-\gamma_1)^{-4} = \frac{3}{(\gamma_2-\gamma_1)^4}, \\ \Omega_{2,2}(\gamma_2,\gamma_1) = \binom{2+3-(2+1)}{3-2}(\gamma_2-\gamma_1)^{-(2+3)+2} = 2.(\gamma_2-\gamma_1)^{-3} = \frac{2}{(\gamma_2-\gamma_1)^3}, \\ \Omega_{2,3}(\gamma_2,\gamma_1) = \binom{2+3-(3+1)}{3-3}(\gamma_2-\gamma_1)^{-(2+3)+3} = 1.(\gamma_2-\gamma_1)^{-2} = \frac{1}{(\gamma_2-\gamma_1)^2}. \end{cases}$$

Since $A_1^{(1)} = (-1)^{2-1} \sum_{\Gamma_{1,1}} \Omega_{1,1}(\gamma_2, \gamma_1)$, $A_1^{(2)} = (-1)^{2-2} \sum_{\Gamma_{1,2}} \Omega_{1,2}(\gamma_2, \gamma_1)$, $A_2^{(1)} = (-1)^{3-1} \sum_{\Gamma_{2,1}} \Omega_{2,1}(\gamma_2, \gamma_1)$, $A_2^{(2)} = (-1)^{3-2} \sum_{\Gamma_{2,2}} \Omega_{2,2}(\gamma_2, \gamma_1)$ and $A_2^{(3)} = (-1)^{3-3} \sum_{\Gamma_{2,3}} \Omega_{2,3}(\gamma_2, \gamma_1)$, we get the corollary.

Corollary 3.7. Let γ_1 and γ_2 be in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with $\gamma_1 \neq \gamma_2$. Then, the partial fraction decomposition of the rational functions $F(x) = \frac{1}{(x-\gamma_1)^2(x-\gamma_2)^3}$ is written under the form,

$$F(x) = \frac{A_1^{(1)}}{x-\gamma_1} + \frac{A_1^{(2)}}{(x-\gamma_1)^2} + \frac{A_2^{(1)}}{x-\gamma_2} + \frac{A_2^{(2)}}{(x-\gamma_2)^2} + \frac{A_2^{(3)}}{(x-\gamma_2)^3}, \tag{14}$$

where

$$\begin{cases} A_1^{(1)} = -\frac{3}{(\gamma_1-\gamma_2)^4}, \ A_1^{(2)} = \frac{1}{(\gamma_1-\gamma_2)^3}, \\ A_2^{(1)} = \frac{3}{(\gamma_2-\gamma_1)^4}, \ A_2^{(2)} = -\frac{2}{(\gamma_2-\gamma_1)^3}, \ A_2^{(3)} = \frac{1}{(\gamma_2-\gamma_1)^2}. \end{cases} \tag{15}$$

The following numerical example is an illustrative application of Expressions Eq. (14)-Eq. (15).

Example 3.8. Consider the rational fraction $F(x) = \frac{1}{(x-8)^2(x-6)^3}$. We show easily that $\gamma_1 = 8$, $\gamma_2 = 6$, $m_1 = 2$ and $m_2 = 3$. Then, application of Expression Eq. (15), allows us to deduce that $A_1^{(1)} = -\frac{3}{16}$, $A_1^{(2)} = \frac{1}{8}$, $A_2^{(1)} = \frac{3}{16}$, $A_2^{(2)} = \frac{1}{4}$ and $A_2^{(3)} = \frac{1}{4}$. Therefore, Expression Eq. (14) implies that the partial fraction decomposition of the rational function $F(x) = \frac{1}{(x-8)^2(x-6)^3}$ is given by,

$$F(x) = -\frac{3}{16}.\frac{1}{x-8} + \frac{1}{8}.\frac{1}{(x-8)^2} + \frac{3}{16}.\frac{1}{x-6} + \frac{1}{4}.\frac{1}{(x-6)^2} + \frac{1}{4}.\frac{1}{(x-6)^3}.$$

This decomposition is consistent with the direct calculation, utilizing the usual methods, which shows the efficiency of Theorem 3.1 and its Corollary 3.7.

Note that, some special cases of polynomials $Q(x)$ have been studied in the literature, but without establishing a compact explicit formulas for the quantities $A_i^{(k)}$.

The generic special case $s = 3$, will be treated in the next section, as a particular case on the general sitting (see Subsection C).

4 General setting and special cases

A General cases Following Theorem 2.1, for every polynomial $R(x)$ of degree $< m_1 + \ldots + m_s$, and $\gamma_i \in \mathbb{K}$ ($1 \leq i \leq s$), with $\gamma_i \neq \gamma_j$, the partial fractional decomposition of the rational fraction $F(x) = \frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}}$ is given by $F(x) = \frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i}\frac{A_i^{(k)}}{(x-\gamma_i)^k}$, where

$$A_i^{(k)} = \frac{1}{(m_i-k)!}\left[\frac{R(x)}{\prod_{j=1,j\neq i}^{s}(x-\gamma_j)^{m_j}}\right]_{x=\gamma_i}^{(m_i-k)}.$$

Application of Theorem 2.2, or more precisely applying the formula Eq. (2), to the function $F_i(x) = R(x)H_i(x)$, permits us to get,

$$F_i^{(d)}(x) = [R \times H_i]^{(d)}(x) = \sum_{k_1+k_2=d}\binom{d}{k_1\, k_2}R^{(k_1)}(x)H_i^{(k_2)}(x),$$

for every $d \geq 0$. Since $R^{(k)} = 0$ for $\deg(R) = p < k \leq d$, the preceding expression takes the form

$$F_i^{(d)}(x) = [R \times H_i]^{(d)}(x) = \sum_{k=0}^{min(d,p)}\binom{d}{k}R^{(k)}(x)H_i^{(d-k)}(x).$$

Therefore, the first amelioration of Theorem 2.1 is given in the following preliminary proposition.

Proposition 4.1. Let $R(x)$ and $Q(x) = \prod_{i=1}^{s}(x-\gamma_i)^{m_i}$ be a polynomials with coefficients in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$); where $0 \leq p = \deg(R) < m_1 + \ldots + m_s$ and $\gamma_i \in \mathbb{K}$ $(1 \leq i \leq s)$, with $\gamma_i \neq \gamma_j$ for $i \neq j$. Then, the partial fraction decomposition of of the rational function $F(x) = \frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}}$, is given by $\frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i}\frac{A_i^{(k)}}{(x-\gamma_i)^k}$, where

$$A_i^{(k)} = \frac{1}{(m_i-k)!}\sum_{h=0}^{min(m_i-k,p)}\binom{m_i-k}{h}R^{(h)}(\gamma_i)H_i^{(m_i-k-h)}(\gamma_i). \tag{16}$$

Now by combining results of Theorem 3.1 and Proposition 4.1, we can establish the compact explicit expression the partial fraction decomposition. That is, the substitution of the expression of $[H_i(x)]_{x=\gamma_i}^{(d_i(k))}$ given by Eq. (6)-Eq. (7) in the formula Eq. (16) of $A_i^{(k)}$, permits to have,

$$A_i^{(k)} = \frac{1}{(m_i-k)!}\sum_{h=0}^{min(m_i-k,p)}\binom{m_i-k}{h}R^{(h)}(\gamma_i)H_i^{(m_i-k-h)}(\gamma_i),$$

or equivalently,

$$A_i^{(k)} = \sum_{h=0}^{min(m_i-k,p)} \sum_{[h_j]_i \in \Gamma_{i,m_i-k-h}} (-1)^{m_i-k-h}(m_i-k-h)! \binom{m_i-k}{h} R^{(h)}(\gamma_i) \Omega_i([h_j]_i;[\gamma_j]),$$

where $[h_j]_i = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_s$, $[\gamma_j] = (\gamma_1, ..., \gamma_s)$ and $\Omega_i([h_j]_i;[\gamma_j])$ is given as in Expression Eq. (7). Therefore, the explicit formula of the partial fraction decomposition is given in the next result.

Theorem 4.2. *Let $Q(x)$ and $R(x)$ be two polynomials such that $Q(x) = \prod_{i=1}^{s}(x-\gamma_i)^{m_i}$, where $\gamma_i \neq \gamma_j$ for $i \neq j$ $(1 \leq i \leq s)$ and* $\deg(R) < m_1 + \ldots + m_s$. *Then, the partial fraction decomposition of $F(x) = \frac{R(x)}{Q(x)}$, is given by,*

$$F(x) = \frac{R(x)}{\prod_{i=1}^{s}(x-\gamma_i)^{m_i}} = \sum_{i=1}^{s}\sum_{k=1}^{m_i}\left[\sum_{d=0}^{min(m_i-k,p)} \sum_{[h_j]_i \in \Gamma_{i,m_i-k-d}} \Psi_{d,i,k}(\gamma_1, ...\gamma_s)\right] \frac{1}{(x-\gamma_i)^k},$$

where $\Gamma_{i,r} = \{[h_j]_i = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_s);\ \sum_{1\leq j\neq i \leq s} h_j = r\}$ and

$$\Psi_{d,i,k}(\gamma_1, ...\gamma_s) = (-1)^{m_i-k-h}(m_i-k-h)! \binom{m_i-k}{h} R^{(h)}(\gamma_i)\Omega_i([h_j]_i;[\gamma_j]), \tag{17}$$

such that $[h_j]_i = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_s$, $[\gamma_j] = (\gamma_1, ..., \gamma_s)$ and $\Omega_i([h_j]_i;[\gamma_j])$ is given by Expression Eq. (7). *In other words, the scalars $A_i^{(k)}$ of partial fraction decomposition of the rational fraction $F(x) = \frac{R(x)}{Q(x)} = \sum_{i=1}^{s}\sum_{k=1}^{m_i}\frac{A_i^{(k)}}{(x-\gamma_i)^k}$, are expressed under the explicit form,*

$$A_i^{(k)} = \sum_{d=0}^{min(m_i-k,p)} \sum_{[h_j]_i \in \Gamma_{i,m_i-k-d}} \Psi_{d,i,k}(\gamma_1, ...\gamma_s), \tag{18}$$

where the $\Psi_{d,i,k}(\gamma_1, ...\gamma_s)$ are given by Eq. (17).

When the degree of the polynomial $R(x)$ is $p = 0$, we derive easily the result of Theorem 3.1. Indeed, for $p = \deg(R) = 0$, we have $R(x) = c$ (constant), thus Expression Eq. (18) takes the form Eq. (9), up to a multiplicative constant by c.

B Special case $s = 2$ and illustrative examples Suppose that $F(x) = \frac{R(x)}{Q(x)} = \frac{R(x)}{(x-\gamma_1)^{m_1}(x-\gamma_2)^{m_2}}$, where $0 \leq p = \deg R < m_1 + m_2$. Then, the partial fraction decomposition of $F(x)$ is given by $F(x) = \sum_{k=1}^{m_1}\frac{A_1^{(k)}}{(x-\gamma_1)^k} + \sum_{k=1}^{m_2}\frac{A_2^{(k)}}{(x-\gamma_2)^k}$.

Let $d_1(k) = \max(m_1 - k, p)$, $d_2(k) = \max(m_2 - k, p)$ and set $\Gamma_{1,d} = \{h_2;\ h_2 = d\}$ for $0 \leq d \leq d_1(k)$ and $\Gamma_{2,d} = \{h_1;\ h_1 = d\}$, for $0 \leq d \leq d_2(k)$. Then, the scalars $A_i^{(k)}$, $i = 1,\, 2$, are given by,

$$A_1^{(k)} = \sum_{d=0}^{min(m_1-k,p)} \sum_{h_2\in\Gamma_{1,m_1-k-d}} (-1)^{m_1-k-d}(m_1-k-d)!\binom{m_1-k}{d} R^{(d)}(\gamma_1)\Omega_1(h_2;[\gamma_j]),$$
$$A_2^{(k)} = \sum_{d=0}^{min(m_2-k,p)} \sum_{h_1\in\Gamma_{2,m_2-k-d}} (-1)^{m_2-k-d}(m_2-k-d)!\binom{m_2-k}{d} R^{(d)}(\gamma_2)\Omega_2(h_1;[\gamma_j]),$$

where $\Omega_1(h_2;[\gamma_j]) = \binom{h_2+m_2-1}{h_2}(\gamma_1-\gamma_2)^{-m_2-h_2}$ and $\Omega_2(h_1;[\gamma_j]) = \binom{h_1+m_1-1}{h_1}(\gamma_2 - \gamma_1)^{-m_1-h_1}$. In summary, we have the following proposition.

Proposition 4.3. Consider the rational fraction $F(x) = \frac{R(x)}{Q(x)}$, where $Q(x)(x-\gamma_1)^{m_1}(x-\gamma_2)^{m_2} =$ with $0 \leq p = \deg R < m_1 + m_2$. Then, the partial fraction decomposition of $F(x)$ is given by $F(x) = \sum_{k=1}^{m_1} \frac{A_1^{(k)}}{(x-\gamma_1)^k} + \sum_{k=1}^{m_2} \frac{A_2^{(k)}}{(x-\gamma_2)^k}$, where

$$A_i^{(k)} = \sum_{d=0}^{min(m_1-k,p)} \sum_{h_2\in\Gamma_{1,m_i-k-d}} (-1)^{m_i-k-d}(m_i-k-d)!\binom{m_i-k}{d} R^{(d)}(\gamma_i)\Omega_i([(h_1,h_2)]_i;[\gamma_j]),$$

for $i = 1,\, 2$, with $\Omega_1(h_2;[\gamma_j]) = \binom{h_2+m_2-1}{h_2}(\gamma_1-\gamma_2)^{-m_2-h_2}$ and $\Omega_2(h_1;[\gamma_j]) = \binom{h_1+m_1-1}{h_1}(\gamma_2-\gamma_1)^{-m_1-h_1}$.

Proposition 4.3 is deduced as a particular case of Theorem 3.1. We will illustrate the steps of its application on the following particular case. Let $F(x) = \frac{ax^2+bx+c}{(x-\gamma_1)^2(x-\gamma_2)}$, where $(a, b) \neq (0, 0)$, $\gamma_1 \neq \gamma_2$. Comparing with the general case, we show that $m_1 = 2$, $m_1 = 1$ and $R(x) = ax^2 + bx + c$, of degree $1 \leq p \leq 2$. Then, we have,

$$F(x) = \frac{ax^2+bx+c}{(x-\gamma_1)^2(x-\gamma_2)} = \frac{A_1^{(1)}}{(x-\gamma_1)} + \frac{A_1^{(2)}}{(x-\gamma_2)^2} + \frac{A_2^{(1)}}{(x-\gamma_1)}. \tag{19}$$

Let exhibit the expressions of $A_1^{(1)}$, $A_1^{(2)}$ and $A_2^{(1)}$ using the compact formula Eq. (18).

Computation of $A_1^{(1)}$ and $A_1^{(2)}$. Following the general case $m_1 = 2$ and $i = 1$, we have

$$A_1^{(k)} = \frac{1}{(m_1-k)!} \sum_{d=0}^{min(m_2-k,1)} \binom{m_1-k}{d} R^{(d)}(\gamma_1) H_1^{(m_1-k-d)}(\gamma_1),$$

where $H_1^{(m_1-k-d)}(\gamma_1) = (-1)^{m_1-k-d}(m_1-k-d)!\Omega_1([h_j]_1;[\gamma_j])$. For $k=1$ we have $m_1-1=1$ and $\min(m_1-1,1)=1$. Thus, we derive,

$$A_1^{(1)} = \frac{1}{1!}\sum_{d=0}^{1}\binom{1}{d}R^{(d)}(\gamma_1)H_1^{(1-d)}(\gamma_1) = R^{(1)}(\gamma_1)H_1^{(0)}(\gamma_1) + R^{(0)}(\gamma_1)H_1^{(1)}(\gamma_1).$$

Since $m_2=1$, $\Gamma_{1,0}=\{h_2;\ h_2=0\}$ and $\Gamma_{1,1}=\{h_2;\ h_2=1\}$, we get,

$$H_1^{(0)}(\gamma_1) = (-1)^0 0!\Omega_1(0,\gamma_1) = \binom{0+m_2-1}{0}(\gamma_1-\gamma_2)^{-m_2} = \frac{1}{\gamma_1-\gamma_2},$$

$$H_1^{(1)}(\gamma_1) = (-1)^1 1!\Omega_1(1,\gamma_1) = \binom{1+m_2-1}{1}(\gamma_1-\gamma_2)^{-m_2-1} = -\frac{1}{(\gamma_1-\gamma_2)^2}.$$

Therefore, it follows that,

$$A_1^{(1)} = R^{(1)}(\gamma_1)H_1^{(0)}(\gamma_1) + R^{(0)}(\gamma_1)H_1^{(1)}(\gamma_1) = \frac{2a\gamma_1+b}{\gamma_1-\gamma_2} - \frac{a\gamma_1^2+b\gamma_1+c}{(\gamma_1-\gamma_2)^2}.$$

Since $R(x)=ax^2+bx+c$ we derive $R(\gamma_1)=a\gamma_1^2+b\gamma_1+c$ and $R^{(1)}(\gamma_1)=2a\gamma_1+b$.

For $k=2$, we show that $m_1-k=2-2=0$, $\min(m_1-2,1)=\min(0,1)=0$ and $\Gamma_{1,0}=\{h_2;\ h_2=0\}$. Hence, we have $A_1^{(2)}(\gamma_1)=\frac{1}{(2-2)!}R^{(0)}(\gamma_1)H_1^{(0)}(\gamma_1)=R^{(0)}(\gamma_1)H_1^{(0)}(\gamma_1)$. Therefore, we derive that $A_1^{(2)}(\gamma_1)=0!(-1)^0R^{(0)}(\gamma_1)\Omega_1(0,\gamma_1)=R(\gamma_1)\binom{0+m_1-1}{0}(\gamma_1-\gamma_2)^{-1}=\frac{R(\gamma_1)}{\gamma_1-\gamma_2}$, and it ensues that,

$$A_1^{(1)} = \frac{a\gamma_1+b}{\gamma_1-\gamma_2} - \frac{a\gamma_1^2+b\gamma_1+c}{(\gamma_1-\gamma_2)^2} \text{ and } A_1^{(2)}(\gamma_1) = \frac{a\gamma_1^2+b\gamma_1+c}{\gamma_1-\gamma_2}. \tag{20}$$

Computation of $A_2^{(1)}$. Following the general case, we show that $m_2=1$ and $i=2$, witch implies that $m_1-1=0$ and $\min(m_1-1,1)=0$ and $\Gamma_{2,0}=\{h_2;\ h_2=0\}$. Hence, we have $A_2^{(1)}=\frac{1}{(1-1)!}R^{(0)}(\gamma_2)H_2^{(0)}(\gamma_2)=R(\gamma_2)H_2^{(0)}(\gamma_2)$, which permits to have $A_2^{(1)}=R(\gamma_2)0!(-1)^0\Omega_2(0;\gamma_2)=R(\gamma_2)\binom{0+m_2-1}{0}(\gamma_2-\gamma_1)^{-m_2-0}=\frac{R(\gamma_2)}{(\gamma_2-\gamma_1)^2}$, namely, we have,

$$A_2^{(1)} = \frac{a\gamma_1^2+b\gamma_1+c}{(\gamma_2-\gamma_1)^2}. \tag{21}$$

In summary, we can formulate the following proposition.

Proposition 4.4. The partial fraction decomposition of $F(x)=\frac{ax+b}{(x-\gamma_1)^2(x-\gamma_2)}$ is given by $F(x)=\frac{A_1^{(1)}}{(x-\gamma_1)}+\frac{A_1^{(2)}}{(x-\gamma_2)^2}+\frac{A_2^{(1)}}{(x-\gamma_1)}$, where $A_1^{(1)}$, $A_1^{(2)}$ and $A_2^{(1)}$ are given as in expressions Eq. (20)-Eq. (21).

To better illustrate the efficiency of the preceding special case, namely, Proposition 4.4, we apply the formulas Eq. (20)-Eq. (21) to the following numerical cases.

Example 4.5. Suppose that $F(x) = \frac{2x+1}{(x-8)^2(x-6)}$. We show easily that $R(x) = 2x+1$, $\gamma_1 = 8$, $\gamma_2 = 6$. Hence, we get $R'(8) = 2$, $\gamma_1 - \gamma_2 = 2$, $R(8) = 17$ and $R(6) = 13$. Now, application of Expressions Eq. (20)-Eq. (21), implies that we have, $A_1^{(1)} = \frac{2}{2} - \frac{17}{4} = -\frac{13}{4}$, $A_1^{(2)} = \frac{17}{2}$ and $A_2^{(1)} = \frac{13}{4}$. Therefore, we get the partial fraction decomposition of the rational function $F(x)$ as follows $F(x) = -\frac{13}{4(x-8)} + \frac{17}{2(x-8)^2} + \frac{13}{4(x-6)}$.

A similar process can be applied to the following numerical example.

Example 4.6. Suppose that $F(x) = \frac{x^2+x+1}{(x-8)^2(x-6)}$. We show easily that $R(x) = x^2+x+1$ and $R'(x) = 2x+1$. Since $\gamma_1 = 8$, $\gamma_2 = 6$, we have $\gamma_1 - \gamma_2 = 2$, $R(8) = 73$, $R'(8) = 17$ and $R(6) = 43$. A direct calculation, applying Expressions Eq. (20)-Eq. (21), shows that $A_1^{(1)} = \frac{17}{2} - \frac{73}{4} = -\frac{39}{4}$, $A_1^{(2)} = \frac{73}{2}$ and $A_2^{(1)} = \frac{43}{4}$. Therefore, the partial fraction decomposition, is given as follows $F(x) = -\frac{39}{4(x-8)} + \frac{73}{2(x-8)^2} + \frac{43}{4(x-6)}$.

Since theses numerical examples are simple, their partial fraction decomposition can be obtained by means of other usual methods. However, our formula Eq. (18) of Theorem 4.2, allows us to compute compact explicit expressions for the coefficients $A_1^{(1)}$, $A_1^{(2)}$ and $A_2^{(1)}$, of the partial fraction decomposition of Expression Eq. (19). And these compact explicit formulas Eq. (20)-Eq. (21), applied to the preceding two numerical examples, allows us to get their associated partial fraction decomposition.

C Special case $s = 3$ and illustrative examples Suppose that $F(x) = \frac{R(x)}{Q(x)} = \frac{R(x)}{(x-\gamma_1)^{m_1}(x-\gamma_2)^{m_2}(x-\gamma_3)^{m_3}}$, where $0 \leq p = \deg R < m_1 + m_2 + m_3$. Then, the partial fraction decomposition of $F(x)$ is given by,

$$F(x) = \sum_{k=1}^{m_1} \frac{A_1^{(k)}}{(x-\gamma_1)^k} + \sum_{k=1}^{m_2} \frac{A_2^{(k)}}{(x-\gamma_2)^k} + \sum_{k=1}^{m_3} \frac{A_3^{(k)}}{(x-\gamma_3)^k}.$$

Let $d_i(k) = \max(m_i - k, p)$, for $i = 1, 2, 3$. For every $0 \leq d \leq d_i(k)$, we set $\Gamma_{1,d} = \{(h_2, h_3);\ h_2 + h_3 = d\}$, $\Gamma_{2,d} = \{(h_1, h_3);\ h_1 + h_3 = d\}$, $\Gamma_{3,d} = \{(h_1, h_2);\ h_1 + h_2 = d\}$. Then, the result of Theorem 3.1, namely, Expression Eq. (18), a direct computation shows that the scalars $A_i^{(k)}$, $i = 1,\ 2,\ 3$, are given by,

$$\begin{aligned} A_1^{(k)} &= \sum_{d=0}^{min(m_1-k,p)} \sum_{(h_2,h_3)\in\Gamma_{1,m_1-k-d}} (-1)^{m_1-k-d}(m_1-k-d)!\binom{m_1-k}{d}R^{(d)}(\gamma_1)\Omega_1((h_2,h_3);[\gamma_j]), \\ A_2^{(k)} &= \sum_{d=0}^{min(m_2-k,p)} \sum_{(h_1,h_3)\in\Gamma_{2,m_2-k-d}} (-1)^{m_2-k-d}(m_1-k-d)!\binom{m_2-k}{d}R^{(d)}(\gamma_2)\Omega_2(h_1,h_3);[\gamma_j]), \\ A_3^{(k)} &= \sum_{d=0}^{min(m_3-k,p)} \sum_{(h_1,h_2)\in\Gamma_{3,m_3-k-d}} (-1)^{m_1-k-d}(m_3-k-d)!\binom{m_3-k}{d}R^{(d)}(\gamma_3)\Omega_3(h_1,h_2);[\gamma_j]), \end{aligned}$$

where the $\Omega_i(h_1,h_2);[\gamma_j])$, for $i = 1,\ 2,\ 3$, are given as in Eq. (7). Accordingly, we can state the following proposition.

Proposition 4.7. The partial fraction decomposition of $F(x) = \dfrac{R(x)}{(x-\gamma_1)^{m_1}(x-\gamma_2)^{m_2}(x-\gamma_3)^{m_3}}$, where $0 \le p = \deg R < m_1+m_2+m_3$ is given by $F(x) = \sum_{k=1}^{m_1} \dfrac{A_1^{(k)}}{(x-\gamma_1)^k} + \sum_{k=1}^{m_2} \dfrac{A_2^{(k)}}{(x-\gamma_2)^k} + \sum_{k=1}^{m_3} \dfrac{A_3^{(k)}}{(x-\gamma_3)^k}$, such that the $A_i^{(k)}$, for $i = 1,\ 2,\ 3$, are given by,

$$A_i^{(k)} = \sum_{d=0}^{min(m_i-k,p)} \sum_{(h_2,h_3)\in\Gamma_{1,m_1-k-d}} \Lambda_{i,d}R^{(d)}(\gamma_1)\Omega_i((h_2,h_3);[\gamma_j]), \tag{22}$$

where $\Lambda_{i,d} = (-1)^{m_i-k-d}(m_i-k-d)!\binom{m_i-k}{d}$ and the $\Omega_i(h_1,h_2);[\gamma_j])$, for $i = 1,\ 2,\ 3$, are given as in Eq. (7), namely,

$$\Omega_i([h_j]_i;[\gamma_j]) = \prod_{j=1,j\neq i}^{3}\left[\binom{h_j+m_j-1}{h_j}(\gamma_i-\gamma_j)^{-m_j-h_j}\right], \text{ for } i = 1,\ 2,\ 3. \tag{23}$$

Suppose that $R(x) = C \in \mathbb{K}$ (constant), then, Expression Eq. (22) takes the form,

$$A_i^{(k)} = (-1)^{m_i-k}(m_i-k)!C \times \Omega_i((h_2,h_3);[\gamma_j]).$$

Example 4.8. Consider the rational fraction $F(x) = \dfrac{2x+1}{(x-8)(x-6)(x-4)}$. We show easily that $R(x) = 2x+1$, $\gamma_1 = 8$, $\gamma_2 = 6$, $\gamma_3 = 4$, which implies that $R(8) = 17$, $R(6) = 13$ and $R(4) = 9$. Now, application of Expressions Eq. (22)-Eq. (23), implies that $A_1^{(1)} = \dfrac{17}{8}$, $A_2^{(1)} = -\dfrac{13}{4}$ and $A_3^{(1)} = \dfrac{9}{8}$. Therefore, the partial fraction decomposition of $F(x)$, is given as follows $F(x) = \dfrac{17}{8(x-8)} - \dfrac{13}{4(x-6)} + \dfrac{9}{8(x-4)}$.

Example 4.9. Suppose that $F(x) = \dfrac{2x+1}{(x-8)(x-6)^2(x-4)}$. We show easily that $R(x) =$

$2x+1$ and $R'(x)=2$, $\gamma_1=8$, $\gamma_2=6$, $\gamma_3=4$, which permits to get $R(8)=17$, $R(6)=13$, $R(4)=9$ and $R'(6)=2$. By applying Expressions Eq. (22)-Eq. (23), we obtain, $A_1^{(1)}=\frac{17}{16}$, $A_2^{(1)}=-\frac{1}{2}$ $A_2^{(2)}=-\frac{13}{4}$ and $A_3^{(1)}=-\frac{9}{16}$. Therefore, the partial fraction decomposition of $F(x)$ is $F(x)=\frac{17}{16(x-8)}-\frac{1}{2(x-6)}-\frac{13}{4(x-6)^2}-\frac{9}{16(x-4)}$.

Example 4.10. Suppose that $F(x)=\frac{2x+1}{(x-8)(x-6)^2(x-4)^3}$. Since $R(x)=2x+1$, $R'(x)=2$ and $R''(x)=0$, $\gamma_1=8$, $\gamma_2=6$, $\gamma_3=4$, we get $R(8)=17$, $R(6)=13$, $R(4)=9$, $R'(8)=2$, $R'(6)=2$, $R'(4)=2$ and $R''(4)=0$. A straightforward computation, utilizing Expressions Eq. (22)-Eq. (23), allows us to obtain $A_1^{(1)}=\frac{17}{256}$, $A_2^{(1)}=\frac{11}{16}$, $A_2^{(2)}=-\frac{13}{16}$, $A_3^{(1)}=-\frac{193}{256}$, $A_3^{(2)}=-\frac{53}{64}$ and $A_3^{(3)}=-\frac{9}{16}$. Therefore, the partial fraction decomposition of $F(x)$ is $F(x)=\frac{17}{256(x-8)}+\frac{11}{16(x-6)}-\frac{13}{16(x-6)^2}-\frac{193}{256(x-4)}-\frac{53}{64(x-4)^2}-\frac{9}{16(x-4)^3}$.

5 Discussion, Concluding remarks and perspective

The partial fraction decomposition of the general case $\frac{R(x)}{Q(x)}$, where the polynomial $Q(x)=\prod_{j=1}^{s}(x-\lambda_j)^{m_j}$, have been largely studied in the literature, by using various methods and techniques. In this study we have proposed an approach to determine the partial fraction decomposition, where there is no need to resort to other techniques. It seems to us that the compact explicit formulas, considered for the calculation of the coefficients $A_i^{(k)}$ $(1\le i\le s,\ 1\le k\le m_j)$, of the partial fractions decomposition, require direct computation. This approach represents another method to determine the partial fractions decomposition, apart from the usual techniques. In addition, comparing with the literature, in the best of our knowledge, we show that there is no explicit formula for the scalars A_i^k.

As a perspective, it seems important to us to highlight these results by a study of an educational nature. Indeed, calculating the coefficients $A_i^{(k)}$, is not an easy task for undergraduate. In the diversity of the usual methods, the choice of the adequate one is not easy for the student. It seems interesting that the compact formulas of this approach, can be proposed to students to allow them to acquire another method to determine the partial fractions decomposition, using other techniques. Moreover, the elaboration of algorithms will allow (as in [9]) to facilitate the use of the software such as MATLAB, which will make it possible to better enhance the content of future educational research, of this new approach.

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