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Peculiarities of smoothly undulating number

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Abstract: This note presents results related to divisibility or multiplicity between two numbers in the class of integers called *smoothly undulating* numbers of the type $uz[n]$. The main result is to characterize and display the types of divisors of some types of numbers $uz[n]$, and we show an algorithm to determine the greatest common divisor between two numbers $uz[n]$.

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1 Introduction

In this work, we present our study on properties related to divisibility criteria in the class of integer numbers called *undulating*. It is observed that throughout the text, the reference to *number* is directed to the elements of the set of non negative integers (natural numbers). For example, if the digits of a number alternate more or less than the digits adjacent to them, such as 3021 and 253612, then the number is called an integer *undulating*. The term *smoothly undulating*, refers to numbers whose adjacent digits alternate only between two digits, as in 7676767 or 858585. For example, see [1–5]. A formal definition is presented below:

Definition 1.1. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of digits in the decimal positional system:

(a) we say that a natural number N, with $n \geq 2$ digits is *undulating* when,

$$N = a_1 a_2 \cdots a_{n-1} a_n, \quad \text{with } a_1 \neq 0 \text{ and } a_i \in D, \text{ for } i = 1, 2, ...n, \tag{1}$$

and alternately $a_1 < a_2$, $a_2 > a_3$, ... or $a_1 > a_2$, $a_2 < a_3$, ..., that is, after the first digit the next digits alternately increase and decrease or decrease and increase. However, the absolute difference values between two adjacent digits can differ.

(b) we say that a natural number N, formed by $n > 2$ digits is *smoothly undulating* when,

$$N = \underbrace{aba \cdots ab}_{n\ even} \text{ or } N = \underbrace{aba \cdots ba}_{n\ odd}, \quad \text{with} \quad a,\ b \in D, \quad a \neq b \text{ and } a \neq 0\ . \tag{2}$$

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Note that the integers 317, 4031, 523265 and 90634360 are *undulating*, while the numbers 3535353 and 9494 are *smoothly undulating*. It is easy to see that in numbers *smoothly undulating* the absolute value of the difference between two adjacent digits is constant.

In the work by Costa and Costa [6, 2021], a study is presented on the primality of numbers *smoothly undulating* and formed only by the digits 1 (one) and 0 (zero), in which it is shown that among these numbers, only number 101 is prime. Already at work, by Carvalho and Costa [7, 2021] presented properties related to base change, divisibility, and primality of undulating numbers. They use list the prime, *smoothly undulating* numbers smaller than 10^{13}.

Here, we will show our study and the results obtained about the numbers *smoothly undulating*. In this context, we consider divisibility criteria already well known in the literature. The main objectives focus on establishing relations between the values a, b and n, according to Definition 1.1(b). We also present properties related to the change of base and divisibility criteria, highlighting the difficulties in finding prime numbers in the class of *smoothly undulating* numbers.

From now on, only the class of *smoothly undulating* numbers will be considered. For simplicity (and convenience), we will denote by AB the set of numbers *smoothly undulating* from Definition 1.1(b) and we will just say that N is a number (with $n > 2$ digits) *smoothly undulating* of type AB, if $N \in AB$.

Example 1.2. The numbers $10101, 232323$ and 5353535 belong to the set AB. Also, the prime numbers $101, 151, 191, 313, 373, 727, 787, 919, 1212121$, and 929292929, among others, are of type AB.

We will use the notation $N = ab[n]$, where n indicates the number of digits in the number $N \in AB$, for $n > 2$. For example, $10[5] = 10101$ and $23[8] = 23232323$. Thus, according to [7], the numbers *smoothly undulating* can be written in the form:

$$ab[n] = \begin{cases} a\sum_{i=0}^{\frac{n-1}{2}} 10^{2i} + b\sum_{i=1}^{\frac{n-1}{2}} 10^{2i-1}, & \text{if } n \text{ is odd,} \\ a\sum_{i=1}^{\frac{n}{2}} 10^{2i-1} + b\sum_{i=0}^{\frac{n}{2}-1} 10^{2i}, & \text{if } n \text{ is even} \quad . \end{cases} \tag{3}$$

This work is structured as follows. In the first two Sections, we make a literature review collecting some results related to the theme, and then we present part of our study involving numbers *gently undulating*. So, in Section 2 of this work, we present some results already known about numbers of the type $ab[n]$. In Section 3 we specify $a = 1$ and $b = 0$, we list some results about numbers of the type $uz[n]$, we present two different proofs of the classic result of the non-existence of prime numbers in the class $uz[n]$ for $n > 3$. In Section 4 we present results related to divisibility or multiplicity between two *smoothly undulating* numbers. In Section 5 we discuss the main results (Theorems 5.2, 5.5, 5.8 and 5.13) of this work and we characterize and display types of divisors of some types of numbers $uz[n]$. In Section 6 we show an algorithm to determine the greatest common divisor between two numbers $uz[n]$. In Section 7, we used the fact that no $uz[n]$ is a perfect square, and furthermore we showed that no $uz[n]$ is a perfect cube.

2 Primality of smoothly undulating numbers

In this Section, our intention is to study the primality of the numbers in the set AB, motivated and inspired by studies and topics already presented (or consecrated) in the literature, related to prime numbers and primes *repunidades* [6–8].

The first results present a characterizations about *smoothly undulating* numbers, occurs when $b = 0$ in $ab[n]$, obtaining the following result:

Proposition 2.1. *[7] No smoothly undulating number $ab[n]$ is prime if $a > 1$ and $b = 0$.*

Proof. Just notice that,

$$a0[n] = a \cdot 10[n] \ .$$

See that $a0[n]$ is a composite number, divisible by a and also by $10[n]$. Therefore, $a0[n]$ is not prime. □

The next presents characterization about the numbers *smoothly undulating* when n is even.

Proposition 2.2. *[2, 3] For any digits a and b, if $n > 3$ is even, then $ab[n]$ is never prime.*

Proof. Considering even $n > 3$, we have $n = 2k$, for some positive integer k. Now, just see that, $ab[n] = ab[2k]$ is of the form,

$$\begin{aligned} ab[2k] &= \underbrace{\underbrace{ab}\,\underbrace{ab}\cdots\underbrace{ab}}_{k \text{ times}} \\ &= ab \cdot 10^{2k-2} + ab \cdot 10^{2k-4} + \cdots + ab \cdot 10^2 + ab \\ &= ab \cdot (10^{2(k-1)} + 10^{2(k-2)} + \cdots + 10^2 + 1) \\ &= ab \cdot 10[2k-1] \ . \end{aligned}$$

Therefore, $ab[n]$ will not be a prime number if $n > 3$ is even. □

A specific case, which we will use later:

Lemma 2.3. *[7] If $n = 6$, then $ab[6] = 3 \cdot 7 \cdot 13 \cdot 37 \cdot ab$, for any $a, b \in D$.*

Proof. It follows from Proposition 2.2 that, $ab[6] = ab \cdot (10[5])$, as $10101 = 3 \cdot 7 \cdot 13 \cdot 37$, it follows that,

$$ab[6] = 3 \cdot 7 \cdot 13 \cdot 37 \cdot ab \ .$$

□

From now on, we consider $ab[n]$ to be numbers of type AB, with n odd. We present other results we obtained about *smoothly undulating* numbers.

Proposition 2.4. *[7] If $a \in \{2, 4, 5, 6, 8\}$ and $n \geq 3$ is odd, then $ab[n]$ is not prime.*

Proof. Just note that if $a \in \{2, 4, 5, 6, 8\}$ and n is odd, then the *smoothly undulating* number $ab[n]$ is even or a multiple of 5. It is not a prime number. □

Before it is worth remembering an auxiliary result, the criterion of divisibility by 3, which can be consulted in [9, 10]:

Lemma 2.5. *An integer n is divisible by 3 if and only if the sum of its digits is a multiple of 3.*

Proposition 2.6. *[7] If a equals 3 or 9 and $n > 3$ is odd, with $n \equiv 1 \mod 3$, then $ab[n]$ is not prime.*

Proof. Since n is odd, it is observed that the number of times the number b appears is $\frac{n-1}{2}$. Since $n-1$ is even and $n-1 \equiv 0 \mod 3$, it follows that $n-1 = 6t$ for some positive integer t, like this

$$\frac{n-1}{2} = \frac{6t}{2} = 3t \ ,$$

that is, $\frac{n-1}{2} \equiv 0 \mod 3$. We also have that, $a \equiv 0 \mod 3$, so the sum of the digits of $ab[n]$ is a multiple of 3. Therefore, by Lemma 2.5, it is concluded that $ab[n]$ is not prime. □

Proposition 2.7. *[7] If $b \in \{3, 6, 9\}$ and $n > 3$ is odd, with $n \equiv 2 \mod 3$, then $ab[n]$ is not prime.*

Proof. If n is odd, the number of times the digit a appears is $\frac{n+1}{2}$. Since $n+1$ is even and $n+1 \equiv 0 \mod 3$, it follows that $n+1 = 6t$, for some t positive integer, like this

$$\frac{n+1}{2} = \frac{6t}{2} = 3t \ ,$$

that is, $a \cdot \frac{n+1}{2} \equiv 0 \mod 3$. We also have that, $b \equiv 0 \mod 3$, so $ab[n]$ is a multiple of 3. Therefore, $ab[n]$ is not prime (Lemma 2.5). □

Proposition 2.8. *[7] For $k \geq 1$, if $n = 3k$ is an odd integer and $a + b + a \equiv 0 \mod 3$, then $ab[n]$ is not prime.*

Proof. The case where $k = 1$, like $a + b + a \equiv 0 \mod 3$, follows that $ab[3]$ is a multiple of 3.

If $n = 3k > 3$ is odd, we have

$$ab[n] = ababab \cdot 10^{n-6} + ababab \cdot 10^{n-12} + \cdots + ababab \cdot 10^{3} + aba \ .$$

It follows from the Corollary 2.3 that, 3 divides the number $ababab$. Therefore, $ab[n]$ is a sum of multiples of 3. Therefore, it is not prime (Lemma 2.5). □

3 Smoothly undulating numbers of the type UZ

Specifying $a = 1$ and $b = 0$ in the Definition 1.1 we have say that a natural number $uz[k]$ is a *smoothly undulating* of the type UZ if it is formed only by the digits 0 and 1 and is an alternating sequence of the digits 0 and 1, starting with 1. For any natural $m \in UZ$, we will use the notation $m = uz[n]$ (general) or $m = 10[n]$ (specific) where $n \geq 2$ indicates the number of digits in the number *smoothly undulating* m. It follows from Proposition 2.1 and Proposition 2.2 that some *smoothly undulating* numbers of type $ab[n]$ are multiples of $uz[n]$, so in the rest of the text we will concentrate our efforts on this class of numbers.

Example 3.1. The numbers $10[2] = 10$, $10[3] = 101$, $10[4] = 1010$, and successively. In particular the number $10[2023] = \underbrace{1010\ldots0101}_{2023}$ with 1012 digits 1 alternating with 1011 digits 0; and more, it follows from the Equation Eq. (3) that

$$10[2023] = 1 \cdot 10^{2022} + 1 \cdot 10^{2020} + \cdots + 1 \cdot 10^2 + 1 \ .$$

Remark 3.2. [6, 11] Note that:

1. if n is even then $uz[n]$ ends in 0, otherwise $uz[n]$ ends in 1.
2. given the natural numbers n, k_1 and k_2 such that $k_1 + k_2 = n$, let $uz[n]$ be a *smoothly undulating* number with k_1 digits 1 alternated by k_2 digits 0 and we write $uz(n) = 1_{k_1}0_{k_2}$. And more, $n = 2k$ is even so the amount, or number of digits, of 1s and 0s in $uz[n]$ are equal and $uz[n] = 1_k0_k$. Whereas if $n = 2k + 1$ is odd then, in $uz[n]$, the quantity of 1s is $k + 1$ and that of 0s is k, that is, in $uz[n] = 1_{k+1}0_k$.

Example 3.3. [6, 11] By direct inspection of the divisors of the first six numbers *smoothly undulating* UZ we obtain that:

1. the number $uz[2] = 10 = 2 \cdot 5$ is not prime;
2. the number $uz[3] = 101$ **is prime**;
3. the number $uz[4] = 1010 = 10 \cdot 101$ is not prime;
4. the number $uz[5] = 10101 = 111 \cdot 91$ is not prime;
5. the number $uz[6] = 101010 = 10 \cdot 10101$ is not prime;
6. the number $uz[7] = 1010101 = 101 \cdot (10^4 + 1)$ is not prime .

In Example 3.3 we list only one factorization, by way of example, when the number $uz[n]$ is composite. In addition to the Example 3.3, we also present Table 1, with the *smoothly undulating* numbers $uz[n]$ and some factors for odd n, with $5 \le n \le 13$.

Table 1. Factors prime

n	$uz[n]$	Factors
5	10101	$3 \cdot 7 \cdot 13 \cdot 37$
7	1010101	$73 \cdot 101 \cdot 137$
9	101010101	$41 \cdot 271 \cdot 9091$
11	10101010101	$3 \cdot 7 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$
13	1010101010101	$239 \cdot 4649 \cdot 909091$

An interesting result about the *smoothly undulating* numbers $uz[n]$ is the following:

Theorem 3.4. *[6, Theorem 5] Except 101, no smoothly undulating number $uz[n]$ is prime.*

Our interest in the primality of these numbers is motivated by reading Ribenboin [8]. Here we will present forward two alternative income statements, different from the one presented in [6].

The next result presents the Binet expression for the *smoothly undulating* numbers

Theorem 3.5. *For every integers $k \geq 2$ we have $uz[n] = \dfrac{10^{2k}-1}{99}$, where $n = 2k-1$.*

Proof. By induction on k, for $k = 2$ we have

$$\begin{aligned} 10[3] =& \frac{10^4-1}{99} = \frac{(10^2-1)(10^2+1)}{(10-1)(10+1)} \\ =& \frac{10^2-1}{10^2-1} \cdot (10^2+1) = 10^2+1 \ , \end{aligned}$$

it follows from the Equation Eq. (3) that $10[3] = 1 \cdot 10^2 + 1$. Assume that for some $k > 2$ we have,

$$\frac{10^{2k}-1}{99} = 1 \cdot 10^{2 \cdot k} + 1 \cdot 10^{2(k-1)} + \cdots + 1 \cdot 10^2 + 1 \ , \tag{4}$$

let's show it for the successor of k. Let's see

$$\begin{aligned} \frac{10^{2(k+1)}-1}{99} =& \frac{10^{2k+2}-1}{99} = \frac{10^{2k+2}-10^2+10^2-1}{99} \\ =& \frac{10^2(10^{2k}-1)}{99} + \frac{10^2-1}{99} \\ \overset{\text{Eq. (4)}}{=}& 10^2 \cdot (1 \cdot 10^{2 \cdot k} + 1 \cdot 10^{2(k-1)} + \cdots + 1 \cdot 10^2 + 1) + 1 \\ =& 1 \cdot 10^{2 \cdot (k+1)} + 1 \cdot 10^{2k} + \cdots + 1 \cdot 10^2 + 1 \ . \end{aligned}$$

Again, it follows from the Equation Eq. (3) that $uz[n+2] = \dfrac{10^{2(k+1)}-1}{99}$, as we wanted. ☐

In the previous proposition, we have that n is odd, in the case where the number of digits is even, it follows directly that

Corollary 3.6. *For every even integers $n \geq 2$ we have $uz[n] = \dfrac{10^{2k}-1}{99} \cdot 10$, where $n = 2k$.*

We will make use of the following auxiliary results, and three results establish conditions on a polynomial factorization, and will be used throughout the work.

Lemma 3.7. *[9, Proposition 3.6] Let a and b be integers and n natural, then $a-b$ divides $a^n - b^n$.*

Lemma 3.8. *[9, Proposition 3.7] Let a and b be integers and n natural, then $a+b$ divides $a^{2n} - b^{2n}$.*

Lemma 3.9. *[9, Proposition 3.8] Let a and b be integers and n natural, then $a+b$ divides $a^{2n+1}+b^{2n+1}$.*

As a direct application of the Lemma 3.7, the next result will display a *smoothly undulating* divisor of a *smoothly undulating* number, in the case where n is composite.

Proposition 3.10. *For any m, $n \in \mathbb{N}$, if m is a multiple of n then $uz[2m-1]$ is a multiple of $uz[2n-1]$.*

Proof. If m is a multiple of n then $m = n \cdot k$ for some integer k . See that, according Theorem 3.5

$$\begin{aligned} uz[2m-1] &= \frac{10^{2m}-1}{99} = \frac{10^{2nk}-1}{99} = \frac{10^{2nk}-1}{10^{2n}-1} \cdot \frac{10^{2n}-1}{99} \\ &= \frac{10^{2nk}-1}{10^{2n}-1} \cdot uz[2n-1] \ . \end{aligned}$$

By Lemma 3.7 we have that $10^{2n}-1$ divides $10^{2nk}-1 = (10^{2n})^k - 1^k$ and thus the *smoothly undulating* $uz[2n-1]$ divides $uz[2m-1]$. □

3.1 First proof of Theorem 3.4

It follows from Corollary 3.6 that the number *smoothly undulating* $uz(2k)$ is a multiple of 10 (even and multiple of 5) and therefore is not prime, justifying the following property.

Proposition 3.11. *No smoothly undulating number $uz[2k]$ is prime.*

Proposition 3.12. *No smoothly undulating number $uz[2k-1]$ is prime, for $k > 2$ integer.*

Proof. According Theorem 3.5 that $uz[2k-1] = \dfrac{10^{2k}-1}{99}$, as

$$uz[2k-1] = \frac{10^{2k}-1}{99} = \frac{(10^k-1)(10^k+1)}{(10-1)(10+1)} \ .$$

We have $k > 2$, if k is odd, then by the Lemma 3.7 we hat that $\dfrac{10^k-1}{10-1}$ is the integer, and by the Lemma 3.9 we hat that $\dfrac{10^k+1}{10+1}$ the same fact.

Now if we have k even and $k = 2^t$ for some $t > 1$ then by the Lemma 3.7 and Lemma 3.8 we hat that $\dfrac{10^k-1}{(10-1)(10+1)}$ is the integer. Finally the case where k is even and $k \neq 2^t$. For some $k > 1$ then $k = t_2 \cdot t_1$, where t_1, t_2 and t_3 are integers, such that $t_1 = 2^{t_3}$ is even and t_1 is odd, so

$$\begin{aligned} uz[2k-1] =& \frac{10^{2k}-1}{99} = \frac{(10^k-1)(10^k+1)}{(10-1)(10+1)} \\ =& \frac{(10^{t_1}-1)(10^{t_1}+1)}{(10-1)(10+1)} \cdot (10^{2t_1}+1)\dots(10^{t_2 \cdot t_1}+1). \end{aligned}$$

Now follows from the first case that $\dfrac{(10^{t_1}-1)(10^{t_1}+1)}{(10-1)(10+1)}$ is integer. ☐

The Theorem 3.4 is a direct consequence of Propositions 3.11 and 3.12.

3.2 Second proof of Theorem 3.4

The numbers repunit are written, in the decimal system, as the repetition of the unit, represented by the set $\mathcal{R}_n = \{1,\ 11,\ 111, \ldots,\ R_n, \ldots\}$. According to [8, 12–14] the equation $R_n = \dfrac{10^n-1}{9}$ presents the Binet expression for the repunit numbers. Soo the Theorem 3.5 presents the Binet expression for the *smoothly undulating* numbers.

Proposition 3.13. *If k is odd and $n = 2k-1$, then $uz[n]$ is a multiple of R_k.*

Proof. Just note that

$$\begin{aligned} uz[n] =& \frac{10^k-1}{9} \cdot \frac{10^k+1}{11} \\ =& R_k \cdot \frac{10^k+1}{11}\ , \end{aligned}$$

like $11 \mid 10^k+1$ if k odd (Lemma 3.9), then the result follows. ☐

Proposition 3.14. *If k is even and $n = 2k-1$, then $uz[n]$ is a multiple of* $10[3]$.

Proof. Just notice that

$$\begin{aligned} uz[n] =& \frac{10^{4k}-1}{99} = \frac{10^4-1}{99} \cdot \frac{10^{4k}-1}{10^4-1} \\ =& 10[3] \cdot \frac{10^{4k}-1}{10^4-1}\ . \end{aligned}$$

The result follows from the fact that $\dfrac{10^{4k}-1}{10^4-1} = \dfrac{(10^4)^k-1}{10^4-1}$ is an integer, according to Lemma 3.7. ☐

Combining Propositions 3.13 and 3.14 it turns out that except 101, no *smoothly undulating* number is prime.

4 Divisibility in UZ

In this Section we will approach some results related to divisibility or multiplicity between two *smoothly undulating* numbers.

From now on we only consider *smoothly undulating* numbers of the type UZ and ending in 1, that is, *smoothly undulating* numbers $uz[n]$ where n is odd, that is, $n = 2k-1$, for some integer k. As $uz[n]$ ends in 1, this is enough to justify the following fact:

Proposition 4.1. *If n is odd then neither 2 nor 5 divides $uz[n]$.*

According to Proposition 4.1 and by definition, if n is odd no $uz[n]$ ends with the digits 0, 2, 4, 6, 8 or 5; that is, no *smoothly undulating* numbers is a multiple of 2 or 5. However, from the Proposition 4.3 we deduce that for any integer n multiple of 3, or multiple of power of 3, there will be an infinite *smoothly undulating* numbers multiples of n. Furthermore, below we will characterize some divisors of *smoothly undulating* numbers.

Proposition 4.2. *[7] If n is odd, with $n \equiv 2 \mod 3$, then $uz[n]$ is multiple of 3.*

Proof. If n is odd then $n+1$ is even and $n+1 \equiv 0 \mod 3$, it follows that $n+1 = 6t$, for some t positive integer, like this

$$\frac{n+1}{2} = \frac{6t}{2} = 3t \ ,$$

that is, $\frac{n+1}{2} \equiv 0 \mod 3$. According Lemma 2.5, we have that, $1 \cdot \frac{n+1}{2} \equiv 0 \mod 3$, so $uz[n]$ is a multiple of 3. □

Proposition 4.3. *If $q = 3^k$ with $k > 0$ integer, then q divides $uz[n]$, where $n = 2q - 1$.*

Proof. Just note that, according Theorem 3.5, Lemma 3.7 and Lemma 2.5,

$$\begin{aligned} uz[n] =& \frac{10^{2q}-1}{99} = \frac{10^{2\cdot 3^k}-1}{99} = \frac{(10^2)^{3^k} - 1^{3^k}}{10^2-1} \\ =& 10^{2(3^k-1)} + 10^{2(3^k-2)} + \cdots + 10^4 + 10^2 + 10^0 \\ \equiv& \underbrace{1+1+1+\cdots+1+1}_{3^k \text{ times}} \equiv 0 \mod 3^k \ . \end{aligned}$$

□

Now it is worth remembering an auxiliary result, the criterion of divisibility by 11, which can be consulted in [9, 10]:

Lemma 4.4. *A number is divisible by 11 if the alternating sum of its digits is divisible by 11.*

Example 4.5. See that R_2 and R_{11} divides $uz[21]$. In fact, according Theorem 3.5,

$$\begin{aligned} uz[21] =& \frac{10^{22}-1}{10^2-1} = \frac{10^{11}-1}{10-1} \cdot \frac{10^{11}+1}{10+1} \\ =& R_{11} \cdot \frac{10^{11}+1}{10+1} \ . \end{aligned}$$

By the Lemma 3.9 we hat that $\frac{10^{11}+1}{10+1}$ is the integer, and

$$\begin{aligned} \frac{10^{11}+1}{10+1} =& 10^{10} - 10^9 + \cdots + 10^2 - 10 + 1 \\ =& 10^9(10-1) + 10^7(10-1) + \cdots + 10^3(10-1) + 10(10-1) + 1 \\ =& 9090909091 \end{aligned}$$

Now follows from Lemma 4.4 that 9090909091 is a multiple of 11. Soon $R_2 = 11$ and R_{11} divides $uz[21]$.

Proposition 4.6. *For all integer $k > 0$, if $n = 11 \cdot k - 1$ then R_2 and R_{11} divide $uz[n]$.*

Proof. According Theorem 3.5,

$$\begin{aligned} uz[n] =& \frac{10^{2(11k)} - 1}{99} = \frac{10^{11k} - 1}{9} \cdot \frac{10^{11k} + 1}{11} \\ =& \left(\frac{10^{11} - 1}{9} \cdot \frac{(10^{11})^k - 1}{10^{11} - 1} \right) \cdot \left(\frac{(10^{11})^k + 1}{10^{11} + 1} \cdot \frac{10^{11} + 1}{11} \right) \\ =& R_{11} \cdot \frac{10^{11} + 1}{11} \cdot \left(\frac{(10^{11})^k - 1}{10^{11} - 1} \cdot \frac{(10^{11})^k + 1}{10^{11} + 1} \right) \ . \end{aligned}$$

In the face of the Lemmas 3.7-3.9, each factor is an integer and follows from the Example 4.5 that $\dfrac{10^{11} + 1}{10 + 1}$ is a multiple of 11. Soon $R_2 = 11$ and R_{11} divides $uz[n]$. □

There is a close relationship between the repunit and *smoothly undulating* numbers, in the Proposition 3.13 we saw that a smoothly undulating number is a multiple of a repunit, now a symmetrical situation

Proposition 4.7. *For every integers $k > 1$, then R_{2k} is a multiple of $uz[n]$, were $n = 2k - 1$.*

Proof. Just note that

$$\begin{aligned} R_{2k} =& 11 \cdot 10^{2k-2} + 11 \cdot 10^{2k-4} + \cdots + 11 \cdot 10^2 + 11 \cdot 10^0 \\ =& 11 \cdot (10^{2k-2} + 10^{2k-4} + \cdots + 10^2 + 10^0) \\ =& 11 \cdot uz[2k - 1] \ . \end{aligned}$$

□

Proposition 4.8. *For every integers $k \geq 1$ we have $uz[5]$ divides $uz[n]$, where $n = 6k - 1$.*

Proof. By induction on k, for $k = 1$ it's trivial.

Assume that for some $k \geq 1$ we have that $uz[5]$ divides $uz[6k - 1]$, that is,

$$uz[6k - 1] = uz[5] \cdot q \ , \tag{5}$$

for some integer q. let's show it for the successor of k. Let's see

$$\begin{aligned} uz[6(k + 1) - 1] &= uz[6k + 5] \\ &= 10^{6k+4} + 10^{6k+2} + 10^{6k} + 10^{6k-2} + \cdot + 10^2 + 10^0 \\ &= 10^{6k} \cdot (10^4 + 10^2 + 10^0) + uz[6k - 1] \\ &= uz[5] \cdot (10^{6k} + q) \end{aligned}$$

In the second addend we use the induction hypothesis, Equation Eq. (5), And we get the result □

Corollary 4.9. *For every integers $k \geq 1$ we have that $uz[n]$ is multiple of 3, 7, 13 and 37; where $n = 6k - 1$.*

Proof. This follows from Proposition 4.8 and that $uz[5] = 3 \cdot 7 \cdot 13 \cdot 37$, looking at Table 1. □

5 Prime factor of smoothly undulating number

To demonstrate the following Theorem 5.2, a well-known result in mathematics will be used, namely the *little Theorem of Fermat*, see [9, 10]:

Lemma 5.1. *If a, $p \in \mathbb{Z}$, with p prime and $(a,p)=1$, then $a^{p-1} \equiv 1 \mod p$.*

Where (a,b) is the greatest common divisor of numbers a and b. As a consequence of the Lemma we have.

Theorem 5.2. *If $p=7$ or $p>11$ is a prime number, then p divides $uz[2p-3]$.*

Proof. According to Fermat's little theorem, $10^{p-1} \equiv 1 \mod p$, or equivalent $(10^{p-1})^2 \equiv 1^2 = 1 \mod p$, with p prime. In particular, $10^{2(p-1)} - 1 \equiv 0 \mod p$. Therefore, p divides $10^{2(p-1)} - 1 = 99 \cdot uz[2p-3]$, as $p=7$ or $p>11$ so $mdc(p,99)=1$ it can be concluded that p necessarily divides $uz[2p-3]$. □

Example 5.3. For the prime $p=7$ we have that 7 divide $uz[11]$, because $n = 2{\cdot}7-3 = 11$. And more, since $11 \equiv 2 \mod 3$, according Proposition 4.2, we have 3 divide $uz[11]$. In fact $uz[11] = 10101010101 = 3 \cdot 7 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$.

For all natural numbers, let $\varphi(m)$ be the number of natural numbers less than or equal to m, that is, $(a,m)=1$ for $a \leq m$. The function $\varphi(m)$ is known as Euler's function. We will make use of the following result, and the proof can be consulted in [9, 10].

Lemma 5.4. *(Euler-Fermat theorem) Let a, m be natural numbers. If $(a,m)=1$, then $a^{\varphi(m)} \equiv 1 \mod m$.*

Now, we consider an integer m that is not a multiple of 3 or 11, with $(2,m)=1$ and $(5,m)=1$, the next result displays a *smoothly undulating* numbers multiple of m, formally we have:

Theorem 5.5. *For any integers $k_1, k_2 \geq 0$ and $q, m > 1$, if $(2,m)=(5,m)=1$ and $m \neq 3^{k_1} \cdot 11^{k_1} \cdot q$, then m divides $uz\big[2\varphi(m)-1\big]$, where $\varphi(m)$ is the Euler function.*

Proof. Let m be a natural $m \neq 3^{k_1} \cdot 11^{k_1} \cdot q$, as $(2,m)=(5,m)=1$ then $(10^2,m) = (10,m) = 1$, since $(2,5)=1$. It follows from the Lemma 5.4 that $10^{\varphi(m)} \equiv 1 \mod m$, or equivalent $(10^{\varphi(m)})^2 \equiv 1^2 \mod m$. Like this

$$99 \cdot uz\big[2\varphi(m)-1\big] = (10^{\varphi(m)})^2 - 1 \equiv 0 \mod m \ .$$

If $m \neq 3^{k_1} \cdot 11^{k_1} \cdot q$ then $(m,99)=1$, that is, m divides $uz\big[2\varphi(m)-1\big]$. □

Let m be a natural number and a any integer such that $(a,m)=1$. Let $h>1$ be the smallest natural number such that $a^h \equiv 1 \mod m$. In this case, we say that the order of a module m is denoted by $ord_m(a)=h$. We will also need the following auxiliary results, additional details and proof can be found at [9, 10].

Lemma 5.6. *If $ord_m(a)=h$, the positive integers k such that $a^k \equiv 1 \mod m$ are precisely those for which $h \mid k$.*

Lemma 5.7. *If* $(a, m) = 1$ *and* $ord_m(a) = h$ *then* h *divides* $\varphi(m)$.

Theorem 5.8. *Let* p, q *be distinct prime numbers with* $p > 2$. *If* q *is a divisor of* $uz[n]$, *then* q *is of the form* $2px + 1$, *for all natural* x, *with* $n = 2p - 1$.

Proof. For all $p > 2$ prime, we have $uz[n] = \dfrac{10^{2p} - 1}{99}$ composite and let q be a prime divisor of $uz[p]$, see that $q \neq 5$, according Proposition 4.1. Since q is a divisor of $uz[n]$ then there is an integer $x_1 > 0$ such that $uz[n] = q \cdot x_1$, hence $99 \cdot (10^{2p} - 1) = 99 \cdot q \cdot x_1$. So $q \mid (10^{2p} - 1)$ where

$$10^{2p} \equiv 1 \mod q \,. \tag{6}$$

Since $(10, q) = 1$, we have $10^{\varphi(q)} \equiv 1 \mod q$. Let $h = ord_q(10)$ and according to Lemma 5.7, we get $h \mid \varphi(q) = q - 1$. It follows from Equation Eq. (6) and Lemma 5.6 that h divides p^2 resulting in h divides the prime p, then $h = p$ and $q = py + 1$ for some natural y, like q is odd entails that $y = 2x$ is even, that is, $q = 2px + 1$ for some natural x. □

Example 5.9. For the prime $p = 7$ we have $n = 2 \cdot 7 - 1 = 13$ and $uz[13] = 1010101010101 = 239 \cdot 4649 \cdot 909091$. See yet that $239 = 2 \cdot 7 \cdot 17 + 1$, $4649 = 2 \cdot 7 \cdot 332 + 1$ and $909091 = 2 \cdot 7 \cdot 64935 + 1$

In Theorem 5.8 it is a characterisation is displayed for a prime factor of a type of *smoothly undulating* number. In addition to the previously presented result, we provide a characterisation for a prime factor of a composite of *smoothly undulating* number. To prove the Theorem 5.13 that will be presented later, we will make use of well-established results about quadratic residues, which can be found in [9, 10, 15].

Definition 5.10. Let $p > 2$ be a prime number and a any integer. We indicate the *Legendre symbol* by:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } p \nmid a \text{ e } a \text{ is residue quadratic modulo } p; \\ -1, \text{ se } p \mid a; \\ 0, \text{ otherwise otherwise} \,. \end{cases}$$

Lemma 5.11. *[Euler's criterion] [9, 15] Let* $p > 2$ *be a prime and* a *be a any integer. So*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \mod p \,.$$

Lemma 5.12. *For every prime* $p > 5$ *we have* $\left(\dfrac{10}{p}\right) = (-1)^{\frac{p-1}{2}}$, *that is,* 10 *is quadratic residue modulo* p *if, and only if,* $p \equiv \pm 1 \mod 6$.

Theorem 5.13. *If* $p > 2$ *is a prime and* $n = 2p - 1$. *Then each prime divisor* $q > 5$ *of* $uz[n]$ *is of the form* $6z \pm 1$, *for some natural* z.

Proof. It follows from Proposition 5.8 that $q = 2px + 1$, with natural x. See that

$$\begin{aligned} 10^{\frac{q-1}{2}} &= 10^{\frac{2px+1-1}{2}} = (10^p)^x \\ &\equiv 1 \mod q \,. \end{aligned}$$

By Euler's criterion, we have $\left(\dfrac{10}{q}\right) = 1$ and 10 is quadratic residue modulo q. Now, it follows from Lemma 5.12 that $q = 6z \pm 1$, for some natural z . □

Such a situation can be described or specified in the following example.

Example 5.14. Again for the prime $p = 7$ we have $n = 2 \cdot 7 - 1 = 13$ and $uz[13] = 1010101010101 = 239 \cdot 4649 \cdot 909091$. From which we notice that $239 = 6 \times 40 - 1$ and $4649 = 6 \times 775 - 1$, and $909091 = 6 \times 151515 + 1$.

6 Greatest common divisor in UZ

Our next result is known as Euclid's Algorithm, and it establishes a method for calculating the greatest common divisor between two numbers, so given the integers a and b we indicate the greatest divisor between them by $gcd(a, b)$, additional details and proof can be found at [9, 10].

Lemma 6.1. *[Euclid's Algorithm] Given the integers a, b, with $b = aq + r$ for q and $0 \leq r < |b|$ integers. The greatest common divisor between a and b is given by $\gcd(a, b) = \gcd(a, r)$.*

As an application of Lemma 6.1 we present the following result:

Proposition 6.2. *For any natural $m \geq n$, with $m,\ n \geq 1$,*

$$\gcd(uz[m], uz[n]) = uz[\gcd(m, n)] \ .$$

Proof. In fact, since $m \geq n$ there are integers $q,\ r_1$ from the Euclidean division of m by n, that is, $m = nq + r_1$ with $0 \leq r_1 < n$. Analogously, let $r_2,\ r_3,\ \ldots, r_s,\ r_{s+1}$ be the partial remainders in Euclid's algorithm. By Lemma 6.1, we have $r_s = (m, n)$. On the other hand, see that $uz[m] = 10^{2r_1} uz[n] + uz[r_1]$, it follows from Lemma 6.1 that

$$\begin{aligned}\gcd(uz[m], uz[n]) &= \gcd(uz[n], uz[r_1]) = \ldots = \gcd(uz[r_s], uz[r_{s+1}]) \\ &= \gcd(uz[r_s], 0) = uz[\gcd(m, n)] \ .\end{aligned}$$

□

Example 6.3. Note that $gcd\Big(uz[100], uz[60]\Big) = uz[20]$, since $gcd(100, 60) = 20$. As well as $gcd\Big(uz[2023], uz[34]\Big) = uz[17]$, since $gcd(2023, 34) = 17$.

Let a and b be any two integers, we say that a is rarely prime with b, which are co-primes, when $(a, b) = 1$.

Corollary 6.4. *Two consecutive smoothly undulating numbers are co-prime.*

Proof. Just note that $gcd(n + 1, n) = 1$ for all natural n. □

We can get the Proposition 3.10 as a direct application of Proposition 6.2, see

Corollary 6.5 (Proposition 3.10)**.** *Let m, n be natural, if n is a multiple of m then $uz[m]$ divides $uz[n]$.*

Proof. As by assumption n is a multiple of m, then there exists an integer q such that $n = mq$, as well as $\gcd(m, n) = \gcd(m, mq) = m$ the result follows from Proposition 6.2. □

7 Smoothly undulating numbers and powers

In this Section we show some results that relate the *smoothly undulating* numbers and powers with a natural exponent.

Proposition 7.1. *The difference between two consecutive smoothly undulating numbers is a perfect square.*

Proof. According to Theorem 3.5 that

$$\begin{aligned} uz[n+1]-uz[n] =& \frac{10^{2n+2}-1}{99} - \frac{10^{2n}-1}{99} = \frac{10^{2n+2}-10^{2n}}{99} \\ =& \frac{10^{2n}(10^2-1)}{99} = (10^n)^2 \ . \end{aligned}$$

□

Corollary 7.2. *If* ab *is a perfect square then the difference* $ab[2n]-ab[2(n-1)]$ *is also a perfect square.*

However, it is known that

Proposition 7.3. *[11] Except of* $uz[1]$*, no other* $uz[n]$ *is a perfect square.*

As consequence we have to

Proposition 7.4. *For* $n \geq 2$ *no* $uz[n]$ *is an even power.*

Proof. Suppose that for some $n \geq 1$ we have $uz[n] = a^{2k}$ with a and $k > 0$ integers, this implies that $uz[n]$ would be a perfect square, considering that $uz[n] = (a^k)^2$, which contradicts the Proposition 7.3. □

Remark 7.5. An open question is whether the *smoothly undulating* numbers are a sum of two powers of even index. Something that doesn't happen in repunit numbers.

Proposition 7.6. *For* $n \geq 2$ *no* $uz[n]$ *is a perfect cube.*

The result will follow with the help of the lemma ahead.

Lemma 7.7. *[16, Theorem 1] Let* $x,\ y,\ m,\ n$ *be natural, with* $x > 1,\ y > 1,\ m > 2,\ n > 1$*, the equation*

$$\frac{x^n-1}{x-1} = y^m \ ,$$

no has a solution $(x,\ y,\ m,\ n)$ *satisfying* $\gcd(x\varphi(x), m) = 1$*, where* $\varphi(x)$ *é the function Euler's action of* x*.*

Now we display a proof for the Proposition 7.6

Proof. According Theorem 3.5, for $n \geq 2$, that

$$uz[n] = \frac{10^{2n}-1}{99} = 1 + 10^2 + 10^4 + 10^6 + \ldots + 10^{2(n-1)} \ .$$

Now, applying the Lemma 7.7, we must show that the Diophantine equation

$$1 + 10^2 + 10^4 + 10^6 + \ldots + 10^{2(n-1)} = y^3 \ ;$$

does not have integer solutions for any n and y, since $mdc(x\varphi(x), m) = 1$, for $x = 10^2$, $\varphi(10^2) = 40$ and $m = 3$. □

8 Considerations

Here we present some results about *smoothly undulating* numbers, specifically those formed alternately by the digits 1 and 0, such a set being denoted by UZ. Furthermore, we show that no element of UZ can be expressed as a perfect cube or as a power of the even index. Moreover, we approach some properties related to the divisibility between two elements of UZ; in particular, we highlight the relationship between the elements of UZ and the repunit numbers. We hope that the presented results can inspire and motivate further studies on this class of numbers.

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