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On the reachability tube of non-Newtonian first-order linear differential equations

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Abstract: A problem of practical interest is the determination of the reachability sets of ordinary differential equations with an external perturbation, or with a control. This problem can be extended to non-Newtonian spaces generated by continuous and injective functions α. This paper presents a method to determine the reachability tube of a family of non-Newtonian first-order linear differential equations with an external perturbation, or with a control, that belongs to a set of functions that are α-continuous and α-bounded. The reachability tube is determined explicitly in three non-Newtonian spaces that are associated with three α-generators. The results obtained are illustrated numerically.

Keywords: α-generators; non-Newtonian calculus; non-Newtonian differential equations; external perturbations; reachability tube.

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1 Introduction

Newtonian calculus emerged in the second half of the 17th century under the contributions of mathematicians such as Isaac Newton, Gottfried Wilhelm Leibnitz, Jakob Bernoulli, Johann Bernoulli, and others, who established the bases of differential and integral calculus, see [5]. The basic operations of Newtonian calculus, integration and differentiation, are the infinitesimal versions of the arithmetic operations on the set of real numbers, addition and subtraction. In this context, the Newtonian calculus is sometimes called an additive calculus, indicating that the basic operation is addition.

The use of additive calculus is quite intuitive when we want to interpret some properties that characterize the set of real numbers. For example, given two real numbers a_1 and b_1, the distance between these numbers is related to the inverse operation of addition: subtraction. In such a case, the distance is represented as the absolute value of the real number $b_1 - a_1$, that is, $|b_1 - a_1|$. The generalization to the case of spaces of greater dimension brings with it a non-essential complication. For example, the distance between two vectors in the plane $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ cannot be interpreted with the argument given in the set of real numbers. The reasoning

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that allows for an interpretation of the distance between these two vectors involves the square root function $\alpha(x) = \sqrt{x}$ and its inverse $\alpha^{-1}(x) = x^2$. In this case, we observe that $\alpha(\alpha^{-1}(b_1 - a_1) + \alpha^{-1}(b_2 - a_2)) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$. According to the definition of Euclidean distance between two vectors, we can consider using $\alpha(\alpha^{-1}(b_1 - a_1) + \alpha^{-1}(b_2 - a_2))$ to define the distance between the vectors $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$, that is,

$$\|\boldsymbol{b} - \boldsymbol{a}\| = \alpha(\alpha^{-1}(b_1 - a_1) + \alpha^{-1}(b_2 - a_2)).$$

Naturally, the extension to more dimensions can be obtained under the same argument; see [12]. On the other hand, this method constitutes a generalization of the one-dimensional case, since if we choose $\boldsymbol{a} = (a_1, 0)$ and $\boldsymbol{b} = (b_1, 0)$, then obviously we obtain the distance on the real line: $\alpha(\alpha^{-1}(b_1 - a_1) + \alpha^{-1}(b_2 - a_2)) = \sqrt{(b_1 - a_1)^2} = |b_1 - a_1|$. The reasoning that has been presented can be used to extend other properties that characterize the set of real numbers. In particular, it is possible to generalize the arithmetic operations and the order that is defined in the set of real numbers. This generates the so-called *non-Newtonian calculus*.

Non-Newtonian calculus was developed in the works of Grossman and Katz in a series of papers which are summarized in [1]. Recently, non-Newtonian calculus has provided a wide variety of mathematical tools for use in science, engineering, and mathematics, and appears to have considerable potential for use as an alternative to the calculus of Newton and Leibniz; see [20, 23, 24]. The basic principles are summarized as follows.

Let X be a non-empty subset of $\mathbb{R}$ and let $\alpha\colon X \to \mathbb{R}$ be an injective function such that the range of this function is a subset $\mathbb{R}_\alpha \subset \mathbb{R}$. The function α is called an α-*generator* of an α-*arithmetic* over $\mathbb{R}_\alpha$ if the following α-*operations* are well defined for each $a, b \in \mathbb{R}_\alpha$:

$$a \overset{\alpha}{\oplus} b = \alpha(\alpha^{-1}(a) + \alpha^{-1}(b)), \qquad \alpha\text{-addition}, \tag{1}$$
$$a \overset{\alpha}{\ominus} b = \alpha(\alpha^{-1}(a) - \alpha^{-1}(b)), \qquad \alpha\text{-substraction}, \tag{2}$$
$$a \overset{\alpha}{\odot} b = \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b)), \qquad \alpha\text{-multiplication}, \tag{3}$$
$$a \overset{\alpha}{\oslash} b = \alpha\left(\frac{\alpha^{-1}(a)}{\alpha^{-1}(b)}\right), \qquad \alpha\text{-division}; \tag{4}$$

see [1]. The set $\mathbb{R}_\alpha$ is called *set of non-Newtonian real numbers*.

The function α allows us to establish an α-*order* on the set $\mathbb{R}_\alpha$ as follows: $a \overset{\alpha}{<} b$ if, and only if, $\alpha^{-1}(a) < \alpha^{-1}(b)$; equivalently: $a \overset{\alpha}{\leq} b$ if, and only if, $\alpha^{-1}(a) \leq \alpha^{-1}(b)$.

In the particular case where $X = \mathbb{R}$ and $\alpha(x) = id(x)$, where $id\colon \mathbb{R} \to \mathbb{R}$ defines the identity function: $id(x) = x$ for each $x \in \mathbb{R}$, the id-operations are reduced to the usual operations on real numbers:

$$a \overset{id}{\oplus} b = a + b, \quad a \overset{id}{\ominus} b = a - b, \quad a \overset{id}{\odot} b = a \cdot b, \quad a \overset{id}{\oslash} b = \frac{a}{b}.$$

We note that, naturally, the id-order coincides with the usual order of real numbers: $a \overset{id}{<} b$ if, and only if, $a < b$.

For other choices of the function α and the set X, an infinite number of α-arithmetic can be obtained, on which Grossman and Katz developed the study of non-Newtonian calculus. We describe two particular cases that are of practical interest.

If we choose the functions $\alpha(x) = \exp(x)$ and $\alpha^{-1}(x) = \ln(x)$ for each $x \in \mathbb{R}$, then the exp-operations (1)–(4) are defined by:

$$a \overset{\exp}{\oplus} b = ab, \quad a \overset{\exp}{\ominus} b = \frac{a}{b}, \quad a \overset{\exp}{\odot} b = a^{\ln(b)} = b^{\ln(a)}, \quad a \overset{\exp}{\oslash} b = a^{\frac{1}{\ln(b)}}, \ \ b \neq 1.$$

The non-Newtonian calculus obtained is called *multiplicative calculus*, or *geometric calculus*. A review of the main properties of the multiplicative calculus can be consulted in [6, 7, 10]. We should mention that this is probably one of the most popular non-Newtonian calculus due to the amount of research and applications reported in the literature. Some applications can be consulted in [12, 19, 20, 23].

On the other hand, if we choose $\alpha(x) = \kappa^{-1}\sinh(x)$ and $\alpha^{-1}(x) = \operatorname{arcsinh}(\kappa x)$ for each $x \in \mathbb{R}$, where $\kappa \in (-1, 1)$, then the corresponding non-Newtonian calculus is called Kaniadakis κ-calculus. The κ-operations (1)–(4) are defined by

$$\begin{aligned}
a \overset{\kappa}{\oplus} b &= \kappa^{-1}\sinh(\operatorname{arcsinh}(\kappa a) + \operatorname{arcsinh}(\kappa b)) = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2},\\
a \overset{\kappa}{\ominus} b &= \kappa^{-1}\sinh(\operatorname{arcsinh}(\kappa a) - \operatorname{arcsinh}(\kappa b)) = x\sqrt{1+\kappa^2 y^2} - y\sqrt{1+\kappa^2 x^2},\\
a \overset{\kappa}{\odot} b &= \kappa^{-1}\sinh\left(\kappa^{-1}\operatorname{arcsinh}(\kappa a)\operatorname{arcsinh}(\kappa b)\right),\\
a \overset{\kappa}{\oslash} b &= \kappa^{-1}\sinh\left(\kappa\frac{\operatorname{arcsinh}(\kappa a)}{\operatorname{arcsinh}(\kappa b)}\right).
\end{aligned}$$

The resulting non-Newtonian calculus describes a generalized form of arithmetic that is generally used in statistical physics; see [8, 9, 15]. Some applications of the Kaniadakis κ-calculus can be consulted in [11, 22, 24].

Other aspects about particular cases of non-Newtonian calculus and their applications can be consulted in [2–4, 21].

The rest of the paper is organized as follows. In Sect. 2 the basic preliminaries of non-Newtonian calculus are presented: its algebraic and topological properties. In Sect. 3 some known properties of non-Newtonian differential calculus and their relationship with Newtonian differential calculus are presented. The problem of finding the reachability tube of the family of non-Newtonian first-order linear differential equations is presented and solved in Sect. 4. The conclusions are formulated in Sect. 5.

2 Preliminaries of non-Newtonian calculus

In this section, we present a brief discussion on the algebraic and topological structure of non-Newtonian real numbers, for which we assume that $X = \mathbb{R}$ and that $\alpha\colon X \to \mathbb{R}_\alpha$ is a continuous and injective function. The results presented are an adaptation of those discussed in [14].

2.1 Algebraic properties of non-Newtonian numbers

One of the main characteristics of non-Newtonian real numbers is that this set is an ordered field.

We remember that an *ordered field* is a system consisting of a set E, four binary operations $(\mathring{\oplus}, \mathring{\ominus}, \mathring{\odot}, \mathring{\oslash})$ defined in E and an order $\mathring{<}$ of the set E. The binary operations defined in E behave in the same way as the binary operations $(+, -, \cdot, /)$ defined in $\mathbb{R}$, and the order of the set E, behaves in the same way as the order $<$ of the set $\mathbb{R}$; see [1]. An ordered field is called an *arithmetic field* when it is a subset of $\mathbb{R}$.

The following results show that $\mathbb{R}_\alpha$ is an ordered field under the operations (1)–(4) and, therefore, is also an arithmetic field.

Theorem 2.1. $(\mathbb{R}_\alpha, \overset{\alpha}{\oplus})$ *is an abelian group.*

Proof. We note that from the definition of α-addition it follows that $a \overset{\alpha}{\oplus} b \in \mathbb{R}_\alpha$ for all $a, b \in \mathbb{R}_\alpha$, that is, the set $\mathbb{R}_\alpha$ is closed under α-addition. On the other hand, if $a, b, c \in \mathbb{R}_\alpha$, then from the definition of the α-addition we observe that

$$\begin{aligned}(a \overset{\alpha}{\oplus} b) \overset{\alpha}{\oplus} c = \alpha(\alpha^{-1}(a \overset{\alpha}{\oplus} b) + \alpha^{-1}(c)) &= \alpha(\alpha^{-1}(\alpha(\alpha^{-1}(a) + \alpha^{-1}(b))) + \alpha^{-1}(c)) \\ &= \alpha(\alpha^{-1}(a) + \alpha^{-1}(b) + \alpha^{-1}(c)) \\ &= \alpha(\alpha^{-1}(a) + \alpha^{-1}(\alpha(\alpha^{-1}(b) + \alpha^{-1}(c)))) \\ &= \alpha(\alpha^{-1}(a) + \alpha^{-1}(b \overset{\alpha}{\oplus} c)) = a \overset{\alpha}{\oplus} (b \overset{\alpha}{\oplus} c).\end{aligned}$$

Therefore, the α-addition is associative. On the other hand, we note that for all $a \in \mathbb{R}_\alpha$: $a \overset{\alpha}{\oplus} \alpha(0) = \alpha(\alpha^{-1}(a)+0) = a = \alpha(0+\alpha^{-1}(a)) = \alpha(0) \overset{\alpha}{\oplus} a$, it follows that $\alpha(0)$ is the zero element of $\mathbb{R}_\alpha$ with respect to the α-addition. Furthermore, for $a \in \mathbb{R}_\alpha$ we observe that $a \overset{\alpha}{\oplus} \alpha(-\alpha^{-1}(a)) = \alpha(\alpha^{-1}(a) - \alpha^{-1}(a)) = \alpha(0) = \alpha(-\alpha^{-1}(a) + \alpha^{-1}(a)) = \alpha(-\alpha^{-1}(a)) \overset{\alpha}{\oplus} a$, that is, $\alpha(-\alpha^{-1}(a)) \in \mathbb{R}_\alpha$ is the additive inverse of $a \in \mathbb{R}_\alpha$. It follows that $(\mathbb{R}_\alpha, \overset{\alpha}{\oplus})$ is a group, and since $a \overset{\alpha}{\oplus} b = \alpha(\alpha^{-1}(a) + \alpha^{-1}(b)) = \alpha(\alpha^{-1}(b) + \alpha^{-1}(a)) = b \overset{\alpha}{\oplus} a$ for all $a, b \in \mathbb{R}_\alpha$, we conclude that this is an abelian group. □

Theorem 2.2. $(\mathbb{R}_\alpha \setminus \{\alpha(0)\}, \overset{\alpha}{\odot})$ *is an abelian group.*

Proof. From the definition of α-multiplication we observe that $a \overset{\alpha}{\odot} b$ is an element of $\mathbb{R}_\alpha \setminus \{\alpha(0)\}$ for all $a, b \in \mathbb{R}_\alpha$, that is, the set $\mathbb{R}_\alpha \setminus \{\alpha(0)\}$ is closed under α-multiplication. Let $a, b, c \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$. It follows from definition of α-multiplication that

$$\begin{aligned}(a \overset{\alpha}{\odot} b) \overset{\alpha}{\odot} c = \alpha(\alpha^{-1}(a \overset{\alpha}{\odot} b) \cdot \alpha^{-1}(c)) &= \alpha(\alpha^{-1}(\alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b))) \cdot \alpha^{-1}(c)) \\ &= \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b) \cdot \alpha^{-1}(c)) \\ &= \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(\alpha(\alpha^{-1}(b) \cdot \alpha^{-1}(c)))) \\ &= \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b \overset{\alpha}{\odot} c)) = a \overset{\alpha}{\odot} (b \overset{\alpha}{\odot} c).\end{aligned}$$

Therefore, α-multiplication is associative. Furthermore, given that for $a \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$ it holds $a \overset{\alpha}{\odot} \alpha(1) = \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(\alpha(1))) = a = \alpha(\alpha^{-1}(\alpha(1)) \cdot \alpha^{-1}(a)) = \alpha(1) \overset{\alpha}{\odot} a$, we conclude that $\alpha(1)$ is the identity of $\mathbb{R}_\alpha \setminus \{\alpha(0)\}$. On the other hand, if $a \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$, then $a \overset{\alpha}{\odot} \alpha(\frac{1}{\alpha^{-1}(a)}) = \alpha(\alpha^{-1}(a) \cdot \frac{1}{\alpha^{-1}(a)}) = \alpha(1) = \alpha(\frac{1}{\alpha^{-1}(a)} \cdot \alpha^{-1}(a)) = \alpha(\frac{1}{\alpha^{-1}(a)}) \overset{\alpha}{\odot} a$, therefore $a \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$ has multiplicative inverse $\alpha(\frac{1}{\alpha^{-1}(a)}) \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$. It follows that $(\mathbb{R}_\alpha \setminus \{\alpha(0)\}, \overset{\alpha}{\odot})$ is a group. We conclude that $\mathbb{R}_\alpha \setminus \{\alpha(0)\}$ is an abelian group, since $a \overset{\alpha}{\odot} b = \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b)) = \alpha(\alpha^{-1}(b) \cdot \alpha^{-1}(a)) = b \overset{\alpha}{\odot} a$ for all $a, b \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$. □

Theorem 2.3. *α-multiplication is distributive over α-addition.*

Proof. Let $a, b, c \in \mathbb{R}_\alpha$. we observe that

$$\begin{aligned} a \overset{\alpha}{\odot} (b \overset{\alpha}{\oplus} c) = \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b \overset{\alpha}{\oplus} c)) &= \alpha(\alpha^{-1}(a) \cdot (\alpha^{-1}(b) + \alpha^{-1}(c))) \\ &= \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b) + \alpha^{-1}(a) \cdot \alpha^{-1}(c)) \\ &= (\alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b))) \overset{\alpha}{\oplus} (\alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(c))) \\ &= (a \overset{\alpha}{\odot} b) \overset{\alpha}{\oplus} (a \overset{\alpha}{\odot} c). \end{aligned}$$

Therefore, the identity $a \overset{\alpha}{\odot} (b \overset{\alpha}{\oplus} c) = (a \overset{\alpha}{\odot} b) \overset{\alpha}{\oplus} (a \overset{\alpha}{\odot} c)$ holds in the set of non-Newtonian real numbers $\mathbb{R}_\alpha$. We can verify that $(b \overset{\alpha}{\oplus} c) \overset{\alpha}{\odot} a = (b \overset{\alpha}{\odot} a) \overset{\alpha}{\oplus} (c \overset{\alpha}{\odot} a)$ by the same method. □

As a consequence of Theorems 2.1, 2.2 and 2.3, we obtain the following result.

Theorem 2.4. $(\mathbb{R}_\alpha, \overset{\alpha}{\oplus}, \overset{\alpha}{\odot}, \overset{\alpha}{<})$ *is an ordered field and therefore is also an arithmetic field.*

The following result shows the relationship that exists between the arithmetic fields $(\mathbb{R}, +, \cdot, <)$ and $(\mathbb{R}_\alpha, \overset{\alpha}{\oplus}, \overset{\alpha}{\odot}, \overset{\alpha}{<})$.

Theorem 2.5. *The fields* $(\mathbb{R}, +, \cdot, <)$ *and* $(\mathbb{R}_\alpha, \overset{\alpha}{\oplus}, \overset{\alpha}{\odot}, \overset{\alpha}{<})$ *are isomorphic.*

Proof. It is enough to see that if $\varphi \colon \mathbb{R} \to \mathbb{R}_\alpha$ is defined by $\varphi(x) = \alpha(x)$, then it holds:

$$\begin{aligned} \varphi(a + b) = \alpha(\alpha^{-1}(\alpha(a)) + \alpha^{-1}(\alpha(b))) = \alpha(a) \overset{\alpha}{\oplus} \alpha(b) = \varphi(a) \overset{\alpha}{\oplus} \varphi(b), \\ \varphi(a \cdot b) = \alpha(\alpha^{-1}(\alpha(a)) \cdot \alpha^{-1}(\alpha(b))) = \alpha(a) \overset{\alpha}{\odot} \alpha(b) = \varphi(a) \overset{\alpha}{\oplus} \varphi(b). \end{aligned}$$

Equivalently, if $\psi \colon \mathbb{R}_\alpha \to \mathbb{R}$ is defined by $\psi(y) = \alpha^{-1}(y)$, then it holds:

$$\begin{aligned} \psi(a \overset{\alpha}{\oplus} b) = \alpha^{-1}(\alpha(\alpha^{-1}(a) + \alpha^{-1}(b))) = \alpha^{-1}(a) + \alpha^{-1}(b) = \psi(a) + \psi(b), \\ \psi(a \overset{\alpha}{\odot} b) = \alpha^{-1}(\alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(b))) = \alpha^{-1}(a) \cdot \alpha^{-1}(b) = \psi(a) \cdot \psi(b). \end{aligned}$$

This shows the result. □

From an algebraic point of view, we observe that some properties of the set of non-Newtonian real numbers can be obtained isomorphically from the corresponding properties of the set of Newtonian real numbers. In the next section we show that this situation can be extended to a topology of the set of non-Newtonian real numbers.

2.2 Topological properties of non-Newtonian numbers

In this section we show that the topology of the set of non-Newtonian real numbers is obtained from the topology of the set of Newtonian real numbers. A more detailed study on these properties can be found in [14, 18].

The *set of non-Newtonian natural numbers* is defined as

$$\mathbb{N}_\alpha = \{\alpha(n) \in \mathbb{R}_\alpha \mid n \in \mathbb{N}\}.$$

We observe that this set is closed under the α-addition, since if $\alpha(n)$ and $\alpha(m)$ are chosen in $\mathbb{N}_\alpha$, then $\alpha(n) \overset{\alpha}{\oplus} \alpha(m) = \alpha(\alpha^{-1}(\alpha(n)) + \alpha^{-1}(\alpha(m))) = \alpha(n + m) \in \mathbb{N}_\alpha$. In particular, it is clear that $\alpha(n) \overset{\alpha}{\oplus} \alpha(1) = \alpha(n + 1)$. Therefore $\alpha(n) = \underbrace{\alpha(1) \overset{\alpha}{\oplus} \cdots \overset{\alpha}{\oplus} \alpha(1)}_{n\text{-times}}$.

If $\alpha(n) \overset{\alpha}{>} \alpha(1)$, then $\alpha(n) \overset{\alpha}{\ominus} \alpha(1) = \alpha(n-1)$. This observation allows us to introduce the *set of non-Newtonian integers*, denoted by $\mathbb{Z}_\alpha$, as the set

$$\mathbb{Z}_\alpha = \{\alpha(n) \in \mathbb{R}_\alpha \mid n \in \mathbb{Z}\}.$$

The set of rational numbers $\mathbb{Q}_\alpha$ is naturally defined as:

$$\mathbb{Q}_\alpha = \{\alpha(n) \overset{\alpha}{\oslash} \alpha(m) \in \mathbb{R}_\alpha \mid \alpha(n), \alpha(m) \in \mathbb{Z}_\alpha \text{ and } \alpha(m) \neq \alpha(0)\}.$$

If $a_1, a_2, \ldots, a_m$ are arbitrary non-Newtonian real numbers, then we use the following notation to denote the sum of these numbers: ${}^\alpha\sum_{k=1}^m a_k = a_1 \overset{\alpha}{\oplus} a_2 \overset{\alpha}{\oplus} \cdots \overset{\alpha}{\oplus} a_m$. Using (1) and induction it can be verified that:

$$\begin{aligned}
a_1 \overset{\alpha}{\oplus} a_2 &= \alpha(\alpha^{-1}(a_1) + \alpha^{-1}(a_1)), \\
a_1 \overset{\alpha}{\oplus} a_2 \overset{\alpha}{\oplus} a_3 &= \alpha(\alpha^{-1}(\alpha(\alpha^{-1}(a_1) + \alpha^{-1}(a_1))) + \alpha^{-1}(a_3)), \\
&= \alpha(\alpha^{-1}(a_1) + \alpha^{-1}(a_1) + \alpha^{-1}(a_3)), \\
&\cdots \\
a_1 \overset{\alpha}{\oplus} a_2 \overset{\alpha}{\oplus} \cdots \overset{\alpha}{\oplus} a_m &= \alpha(\alpha^{-1}(a_1) + \alpha^{-1}(a_2) + \cdots + \alpha^{-1}(a_m)),
\end{aligned}$$

that is,

$${}^\alpha\sum_{k=1}^m a_k = \alpha\left(\sum_{k=1}^m \alpha^{-1}(a_k)\right).$$

For each $a \in \mathbb{R}_\alpha$ and each $\alpha(m) \in \mathbb{N}_\alpha$, we denote by $a^{\alpha(m)}$ the $\alpha(m)$-*th power* of a, which is obtained by α-multiplying m-times the number a with itself:

$$\begin{aligned}
a^{\alpha(2)} &= a \overset{\alpha}{\odot} a = \alpha(\alpha^{-1}(a) \cdot \alpha^{-1}(a)) = \alpha((\alpha^{-1}(a))^2), \\
a^{\alpha(3)} &= a^{\alpha(2)} \overset{\alpha}{\odot} a = \alpha(\alpha^{-1}(\alpha((\alpha^{-1}(a))^2)) \cdot \alpha^{-1}(a)) = \alpha((\alpha^{-1}(a))^3), \\
&\cdots \\
a^{\alpha(m)} &= a^{\alpha(m-1)} \overset{\alpha}{\odot} a = \alpha((\alpha^{-1}(a))^m),
\end{aligned}$$

that is,

$$a^{\alpha(m)} = \alpha((\alpha^{-1}(a))^m), \quad m \in \mathbb{N}. \tag{5}$$

We observe from (5) that $a^{\alpha(1)} = \alpha(\alpha^{-1}(a)) = a$ and that $a^{\alpha(0)} = \alpha(1)$. We also observe that $a^{\alpha(-1)} = \alpha(\frac{1}{\alpha^{-1}(a)})$, that is, $a^{\alpha(-1)}$ is the multiplicative inverse of a, see Theorem 2.2. Therefore, we conclude that: $a \overset{\alpha}{\odot} a^{\alpha(-1)} = a^{\alpha(-1)} \overset{\alpha}{\odot} a = \alpha(1)$ for all $a \in \mathbb{R} \setminus \{\alpha(0)\}$.

An application that is obtained as a consequence of (5) is that for each $a \in \mathbb{R}_\alpha$ and each $m \in \mathbb{N}$, the equation $b^{\alpha(m)} = a$ has the solution

$$b = \alpha\Big(\sqrt[m]{\alpha^{-1}(a)}\Big),$$

which is verified directly by substituting in the corresponding equation. This allows us to introduce the $\alpha(m)$-*th root* of a non-Newtonian real number $a \in \mathbb{R}_\alpha$, which is denoted by $\sqrt[m]{a}^{\,\alpha}$ and is defined as the number

$$\sqrt[m]{a}^{\,\alpha} = \alpha\Big(\sqrt[m]{\alpha^{-1}(a)}\Big). \tag{6}$$

As a particular case, we observe that the *α-square root* of $a \in \mathbb{R}_\alpha$ is defined as

$$\sqrt{a}^{\alpha} = \alpha\Big(\sqrt{\alpha^{-1}(a)}\Big).$$

In the same way, we introduce the *α-absolute value* of a non-Newtonian real number $a \in \mathbb{R}_\alpha$, which is denoted by $|a|_\alpha$ and is defined as

$$|a|_\alpha = \sqrt{a^{\alpha(2)}}^{\alpha} = \alpha\Big(\sqrt{\alpha^{-1}(\alpha((\alpha^{-1}(a))^2))}\Big) = \alpha(|\alpha^{-1}(a)|),$$

that is,

$$|a|_\alpha = \alpha(|\alpha^{-1}(a)|). \tag{7}$$

If we consider the possible cases for $a \in \mathbb{R}_\alpha$, we obtain as a consequence of the representation (7), the definition of the function *α-absolute value* $|\cdot|_\alpha \colon \mathbb{R}_\alpha \to \mathbb{R}_\alpha$, which is defined as

$$|a|_\alpha = \begin{cases} a, & \text{if } a \overset{\alpha}{>} \alpha(0), \\ \alpha(0), & \text{if } a = \alpha(0), \\ \alpha(0) \overset{\alpha}{\ominus} a, & \text{if } a \overset{\alpha}{<} \alpha(0). \end{cases} \tag{8}$$

The particularity of the representations (7)–(8) is due to the fact that the α-absolute value allows us to characterize the non-Newtonian real numbers $\mathbb{R}_\alpha$. Let $a \in \mathbb{R}_\alpha$. We say that a is a *positive non-Newtonian real number* if it satisfies $a \overset{\alpha}{\geq} \alpha(0)$, a is a *negative non-Newtonian real number* if it satisfies $a \overset{\alpha}{\leq} \alpha(0)$ and, finally, we say that a is an *unsigned non-Newtonian real number* if it satisfies $a = \alpha(0)$. Therefore, the set of non-Newtonian real numbers $\mathbb{R}_\alpha$ admits the decomposition $\mathbb{R}_\alpha = \mathbb{R}_\alpha^+ \cup \mathbb{R}_\alpha^- \cup \{\alpha(0)\}$, where

$$\mathbb{R}_\alpha^+ = \{a \in \mathbb{R}_\alpha \mid a \overset{\alpha}{>} \alpha(0)\}, \quad \mathbb{R}_\alpha^- = \{a \in \mathbb{R}_\alpha \mid a \overset{\alpha}{<} \alpha(0)\}.$$

We call $\mathbb{R}_\alpha^+$ the *set of positive non-Newtonian real numbers* and $\mathbb{R}_\alpha^-$ the *set of negative non-Newtonian real numbers*. It is clear from the above definitions that $\mathbb{N}_\alpha \subset \mathbb{R}_\alpha^+$. Furthermore, we call the set $\bar{\mathbb{R}}_\alpha^+ = \mathbb{R}_\alpha^+ \cup \{\alpha(0)\}$ the *set of non-negative non-Newtonian real numbers*.

A property of the absolute value in the set of real numbers is that $|b_1 \cdot b_2| = |b_1| \cdot |b_2|$ for each $b_1, b_2 \in \mathbb{R}$. This property also holds for the set of non-Newtonian real numbers with respect to α-absolute value. To see that this is so, let $b_1, b_2 \in \mathbb{R}_\alpha$. Then it follows that

$$\begin{aligned} |b_1 \overset{\alpha}{\odot} b_2|_\alpha = \alpha(|\alpha^{-1}(b_1 \overset{\alpha}{\odot} b_2)|) &= \alpha\Big(\Big|\alpha^{-1}(\alpha(\alpha^{-1}(b_1) \cdot \alpha^{-1}(b_2)))\Big|\Big) \\ &= \alpha(|\alpha^{-1}(b_1) \cdot \alpha^{-1}(b_2)|) \\ &= \alpha(|\alpha^{-1}(b_1)| \cdot |\alpha^{-1}(b_2)|) \\ &= \alpha(\alpha^{-1}(\alpha(|\alpha^{-1}(b_1)|)) \cdot \alpha^{-1}(\alpha(|\alpha^{-1}(b_2)|))) \\ &= \alpha(|\alpha^{-1}(b_1)|) \overset{\alpha}{\odot} \alpha(|\alpha^{-1}(b_2)|) = |b_1|_\alpha \overset{\alpha}{\odot} |b_2|_\alpha, \end{aligned}$$

that is,

$$|b_1 \overset{\alpha}{\odot} b_2|_\alpha = |b_1|_\alpha \overset{\alpha}{\odot} |b_2|_\alpha. \tag{9}$$

Another property of the absolute value in the set of real numbers is the triangular inequality: $|b_1 + b_2| \leq |b_1| + |b_2|$ for each $b_1, b_2 \in \mathbb{R}$. This property also holds for the set of non-Newtonian real numbers. To see that this is so, let $b_1, b_2 \in \mathbb{R}_\alpha$. Then we observe from (7) that

$$|b_1 \overset{\alpha}{\oplus} b_2|_\alpha = \alpha(|\alpha^{-1}(b_1 \overset{\alpha}{\oplus} b_2)|) = \alpha(|\alpha^{-1}(\alpha(\alpha^{-1}(b_1)+\alpha^{-1}(b_2)))|) = \alpha(|\alpha^{-1}(b_1)+\alpha^{-1}(b_2)|).$$

Therefore, if we apply α^{-1} on both sides of this equality, we obtain

$$\alpha^{-1}(|b_1 \overset{\alpha}{\oplus} b_2|_\alpha) = |\alpha^{-1}(b_1) + \alpha^{-1}(b_2)| \leq |\alpha^{-1}(b_1)| + |\alpha^{-1}(b_2)|,$$

and, since α is injective, we obtain

$$\begin{aligned} |b_1 \overset{\alpha}{\oplus} b_2|_\alpha \overset{\alpha}{\leq} \alpha(|\alpha^{-1}(b_1)| + |\alpha^{-1}(b_2)|) &\overset{\alpha}{\leq} \alpha(\alpha^{-1}(\alpha(|\alpha^{-1}(b_1)|)) + \alpha^{-1}(\alpha(|\alpha^{-1}(b_2)|))) \\ &= \alpha(|\alpha^{-1}(b_1)|) \overset{\alpha}{\oplus} \alpha(|\alpha^{-1}(b_1)|) = |b_1|_\alpha \overset{\alpha}{\oplus} |b_2|_\alpha. \end{aligned}$$

Therefore, for all $b_1, b_2 \in \mathbb{R}_\alpha$ we conclude that:

$$|b_1 \overset{\alpha}{\oplus} b_2|_\alpha \overset{\alpha}{\leq} |b_1|_\alpha \overset{\alpha}{\oplus} |b_2|_\alpha. \tag{10}$$

The *distance* between two numbers $a_1, a_2 \in \mathbb{R}_\alpha$ is defined as $|a_1 \overset{\alpha}{\ominus} a_2|_\alpha$. We observe that this distance is an element of $\bar{\mathbb{R}}^+_\alpha$, which follows from the representation (7), and from observing that:

$$\begin{aligned} |a_1 \overset{\alpha}{\ominus} a_2|_\alpha = \alpha(|\alpha^{-1}(a_1 \overset{\alpha}{\ominus} a_2)|) &= \alpha(|\alpha^{-1}(\alpha(\alpha^{-1}(a_1) - \alpha^{-1}(a_2)))|) \\ &= \alpha(|\alpha^{-1}(a_1) - \alpha^{-1}(a_2)|), \end{aligned}$$

that is,

$$|a_1 \overset{\alpha}{\ominus} a_2|_\alpha = \alpha(|\alpha^{-1}(a_1) - \alpha^{-1}(a_2)|). \tag{11}$$

We observe that the inequality $|a_1 \overset{\alpha}{\ominus} a_2|_\alpha \overset{\alpha}{\geq} \alpha(0)$ is satisfied, since this is equivalent to $|\alpha^{-1}(a_1) - \alpha^{-1}(a_2)| \geq 0$. Furthermore, it is clear that $|a_1 \overset{\alpha}{\ominus} a_2|_\alpha = \alpha(0)$ if, and only if, $\alpha^{-1}(a_1) = \alpha^{-1}(a_2)$. On the other hand, it follows that the distance is symmetric, in the sense that $|a_1 \overset{\alpha}{\ominus} a_2|_\alpha = \alpha(|\alpha^{-1}(a_1) - \alpha^{-1}(a_2)|) = \alpha(|\alpha^{-1}(a_2) - \alpha^{-1}(a_1)|) = |a_2 \overset{\alpha}{\ominus} a_1|_\alpha$. Finally, for each $a_1, a_2, a_3 \in \mathbb{R}_\alpha$ it holds $|a_1 \overset{\alpha}{\ominus} a_3|_\alpha \overset{\alpha}{\leq} |a_1 \overset{\alpha}{\ominus} a_2|_\alpha \overset{\alpha}{\oplus} |a_2 \overset{\alpha}{\ominus} a_3|_\alpha$, which is obtained from (10), and from observing that if we choose $b_1 = a_1 \overset{\alpha}{\ominus} a_2$ and $b_2 = a_2 \overset{\alpha}{\ominus} a_3$, then

$$\begin{aligned} b_1 \overset{\alpha}{\oplus} b_2 = (a_1 \overset{\alpha}{\ominus} a_2) \overset{\alpha}{\oplus} (a_2 \overset{\alpha}{\ominus} a_3) &= \alpha(\alpha^{-1}(a_1 \overset{\alpha}{\ominus} a_2) + \alpha^{-1}(a_2 \overset{\alpha}{\ominus} a_3)) \\ &= \alpha(\alpha^{-1}(a_1) - \alpha^{-1}(a_2) + \alpha^{-1}(a_2) - \alpha^{-1}(a_3)) \\ &= \alpha(\alpha^{-1}(a_1) - \alpha^{-1}(a_3)) = a_1 \overset{\alpha}{\ominus} a_3. \end{aligned}$$

As a consequence, the function $\varrho_\alpha \colon \mathbb{R}_\alpha \times \mathbb{R}_\alpha \to \mathbb{R}_\alpha$ defined by

$$\varrho_\alpha(a_1, a_2) = |a_1 \overset{\alpha}{\ominus} a_2|_\alpha$$

induces a metric on $\mathbb{R}_\alpha$, which satisfies for all $a_1, a_2, a_3 \in \mathbb{R}_\alpha$ the following conditions:

1. $\varrho_\alpha(a_1, a_2) \overset{\alpha}{\geq} \alpha(0)$.

2. $\varrho_\alpha(a_1, a_2) = \alpha(0)$ if, and only if, $a_1 = a_2$.

3. $\varrho_\alpha(a_1, a_2) = \varrho_\alpha(a_2, a_1)$.

4. $\varrho_\alpha(a_1, a_3) \overset{\alpha}{\leq} \varrho_\alpha(a_1, a_2) \overset{\alpha}{\oplus} \varrho_\alpha(a_2, a_3)$.

The following result is obtained:

Theorem 2.6. *The non-Newtonian space $\mathbb{R}_\alpha$ is a metric space with metric ϱ_α defined by*

$$\varrho_\alpha(a_1, a_2) = |a_1 \overset{\alpha}{\ominus} a_2|_\alpha = \alpha\Big(|\alpha^{-1}(a_1) - \alpha^{-1}(a_2)|\Big). \tag{12}$$

The metric space $(\mathbb{R}_\alpha, \varrho_\alpha)$ is called *non-Newtonian metric space* and ϱ_α is called *non-Newtonian metric.*

We observe that if in $\mathbb{R}$ we consider the standard metric $\varrho\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\varrho(b_1, b_2) = |b_1 - b_2|$, then

$$\varrho_\alpha(a_1, a_2) = \alpha\Big(\varrho(\alpha^{-1}(a_1), \alpha^{-1}(a_2))\Big), \quad a_1, a_2 \in \mathbb{R}_\alpha, \tag{13}$$

which is obtained from (11). This property allows us to observe that the topology $\mathcal{T}_\alpha$ for $\mathbb{R}_\alpha$, which is induced by the metric ϱ_α, is obtained from the topology $\mathcal{T}$ for $\mathbb{R}$ induced by the metric ϱ.

Let a and b be two arbitrary non-Newtonian real numbers. We denote by $(a, b)_\alpha$ the set of non-Newtonian real numbers $x \in \mathbb{R}_\alpha$ that satisfy the following inequalities $a \overset{\alpha}{<} x \overset{\alpha}{<} b$, that is, the open interval $(a, b)_\alpha = \{x \in \mathbb{R}_\alpha \mid a \overset{\alpha}{<} x \overset{\alpha}{<} b\}$. Equivalently, we identify this set with the set of real numbers $\alpha^{-1}(x) \in \mathbb{R}$ that satisfy $\alpha^{-1}(a) < \alpha^{-1}(x) < \alpha^{-1}(b)$, that is, with the open interval of real numbers $(\alpha^{-1}(a), \alpha^{-1}(b))$. The above allows us to express an *open interval of non-Newtonian real numbers* as

$$(a, b)_\alpha = \alpha\Big((\alpha^{-1}(a), \alpha^{-1}(b))\Big). \tag{14}$$

Similarly, we denote by $[a, b]_\alpha = \{x \in \mathbb{R}_\alpha \mid a \overset{\alpha}{\leq} x \overset{\alpha}{\leq} b\}$ a closed interval of non-Newtonian real numbers. Under an argument similar to the one used, we express a *closed interval of non-Newtonian real numbers* as

$$[a, b]_\alpha = \alpha\Big([\alpha^{-1}(a), \alpha^{-1}(b)]\Big). \tag{15}$$

Let $a \in \mathbb{R}_\alpha$ be arbitrary and $\epsilon \in \mathbb{R}_\alpha$ such that $\epsilon \overset{\alpha}{>} \alpha(0)$. An *open non-Newtonian ball of center a and radius ϵ*, which is denoted by $B_\alpha(a, \epsilon)$, is defined as the set of non-Newtonian real numbers $x \in \mathbb{R}_\alpha$ such that $\varrho_\alpha(x, a) \overset{\alpha}{<} \epsilon$, that is, $B_\alpha(a, \epsilon) = \{x \in \mathbb{R}_\alpha \mid \varrho_\alpha(x, a) \overset{\alpha}{<} \epsilon\}$. According to (7) and (11), we observe that if $x \in B_\alpha(a, \epsilon)$, then $\alpha(|\alpha^{-1}(x \overset{\alpha}{\ominus} a)|) \overset{\alpha}{<} \epsilon$, and as $\alpha^{-1}(x \overset{\alpha}{\ominus} a) = \alpha^{-1}(x) - \alpha^{-1}(a)$, then we can express the last inequality in the following equivalent form $\alpha^{-1}(a) - \alpha^{-1}(\epsilon) < \alpha^{-1}(x) < \alpha^{-1}(a) + \alpha^{-1}(\epsilon)$. As a consequence, we can express a non-Newtonian open ball with center $a \in \mathbb{R}_\alpha$ and radius $\epsilon \overset{\alpha}{>} \alpha(0)$ as

$$B_\alpha(a, \epsilon) = \alpha\Big(B(\alpha^{-1}(a), \alpha^{-1}(\epsilon))\Big), \tag{16}$$

where $B(\alpha^{-1}(a), \alpha^{-1}(\epsilon))$ is a open ball of center $\alpha^{-1}(a) \in \mathbb{R}$ and radius $\alpha^{-1}(\epsilon) > 0$, that is, if $B(\alpha^{-1}(a), \alpha^{-1}(\epsilon))$ represents the open interval $(\alpha^{-1}(a) - \alpha^{-1}(\epsilon), \alpha^{-1}(a) + \alpha^{-1}(\epsilon))$.

Following the same reasoning, we observe that if the set $\bar{B}(\alpha^{-1}(a), \alpha^{-1}(\epsilon))$ describes a closed ball of center $\alpha^{-1}(a) \in \mathbb{R}$ and radius $\alpha^{-1}(\epsilon) > 0$, that is, if $\bar{B}(\alpha^{-1}(a), \alpha^{-1}(\epsilon))$ represents the closed interval $[\alpha^{-1}(a) - \alpha^{-1}(\epsilon), \alpha^{-1}(a) + \alpha^{-1}(\epsilon)]$, then we can express a *closed non-Newtonian ball of center a and radius ϵ* as

$$\bar{B}_\alpha(a, \epsilon) = \alpha\Big(\bar{B}(\alpha^{-1}(a), \alpha^{-1}(\epsilon))\Big). \tag{17}$$

The expressions (16) and (17) allow us to observe that there is a dependence between the topologies of $\mathbb{R}$ and $\mathbb{R}_\alpha$. We recall that a basis for a topology on $\mathbb{R}$ is a family $\mathcal{B}$ of open sets such that each open subset of $\mathbb{R}$ is expressed as a union of some members of $\mathcal{B}$; see [17]. In particular, it is well known that if $\mathcal{B} = \{B(a, \epsilon) \mid a \in \mathbb{R} \text{ and } \epsilon > 0\}$, then $\mathcal{B}$ is a basis for the usual $\mathcal{T}$ topology of $\mathbb{R}$, which is induced by the usual metric ϱ on $\mathbb{R}$. Furthermore, it is well known that if α is a continuous function, then α^{-1} is also a continuous function, since α is injective. We observe that from this it follows that

$$\mathcal{B}_\alpha = \{\alpha(B(a, \epsilon)) \mid B(a, \epsilon) \in \mathcal{B}\}, \tag{18}$$

is a basis for a topology $\mathcal{T}_\alpha$ for $\mathbb{R}_\alpha$. This allows us to obtain the following result.

Theorem 2.7. *Let $\mathcal{B}_\alpha$ be defined as in (18). Then $\mathcal{B}_\alpha$ is a basis for a topology $\mathcal{T}_\alpha$ for $\mathbb{R}_\alpha$ that is induced by the function α from the usual topology $\mathcal{T}$ of the set of real numbers $\mathbb{R}$.*

3 Preliminaries of non-Newtonian differential calculus

The importance of the Theorem 2.7 allows us to reformulate the basic concepts for functions of a variable in non-Newtonian calculus: α-limit, α-continuity, α-differentiability, etc. These concepts allow us to introduce the basic principles of non-Newtonian differential calculus.

Let $f\colon \mathbb{R}_\alpha \to \mathbb{R}_\alpha$ be a function and $x \in \mathbb{R}_\alpha$.[i] We say that $\ell \in \mathbb{R}_\alpha$ is the α-*limit* of the function f as x tends to $a \in \mathbb{R}_\alpha$, which is denoted by

$$^{\alpha}\lim_{x\to a} f(x) = \ell, \tag{19}$$

if for all $\epsilon \overset{\alpha}{>} \alpha(0)$ there exists a non-Newtonian real number $\delta \overset{\alpha}{>} \alpha(0)$ such that if $\alpha(0) \overset{\alpha}{<} \varrho_\alpha(x, a) \overset{\alpha}{<} \delta$ then $\varrho_\alpha(f(x), \ell) \overset{\alpha}{<} \epsilon$. Equivalently, $\ell \in \mathbb{R}_\alpha$ is the non-Newtonian limit of the function f as x tends to $a \in \mathbb{R}_\alpha$, if for all $\epsilon \overset{\alpha}{>} \alpha(0)$ exists a non-Newtonian real number $\delta \overset{\alpha}{>} \alpha(0)$ such that if $x \in B_\alpha(a, \delta) \setminus \{a\}$, then $f(x) \in B_\alpha(\ell, \epsilon)$.

From the definition of α-limit of a function, we can verify that if $^{\alpha}\lim_{x\to a} f_1(x) = \ell_1$ and $^{\alpha}\lim_{x\to a} f_2(x) = \ell_2$, then

$$^{\alpha}\lim_{x\to a}(f_1(x) \overset{\alpha}{\oplus} f_2(x)) = \ell_1 \overset{\alpha}{\oplus} \ell_2, \quad ^{\alpha}\lim_{x\to a}(f_1(x) \overset{\alpha}{\ominus} f_2(x)) = \ell_1 \overset{\alpha}{\ominus} \ell_2,$$
$$^{\alpha}\lim_{x\to a}(f_1(x) \overset{\alpha}{\odot} f_2(x)) = \ell_1 \overset{\alpha}{\odot} \ell_2,$$

and if $\ell_2 \neq \alpha(0)$, then

$$^{\alpha}\lim_{x\to a}(f_1(x) \overset{\alpha}{\oslash} f_2(x)) = \ell_1 \overset{\alpha}{\oslash} \ell_2.$$

[i] We can also consider functions $f\colon \mathbb{R}_\alpha \to \mathbb{R}_\sigma$, where $\mathbb{R}_\alpha$ is a set of non-Newtonian real numbers generated by a function α and $\mathbb{R}_\sigma$ is a set of non-Newtonian real numbers generated by a function σ. Some general cases can be consulted in [16].

According to the previous definitions, we can establish the meaning of the expression ${}^{\alpha}\lim_{y\to x} f(y) = f(x)$ and introduce the concept of α-continuity of the function f at a point $x \in \mathbb{R}_\alpha$. We say that f is *α-continuous* in $x \in \mathbb{R}_\alpha$, if for all $\epsilon \overset{\alpha}{>} \alpha(0)$ there exists a non-Newtonian real number $\delta \overset{\alpha}{>} \alpha(0)$ such that if $\varrho_\alpha(x, z) \overset{\alpha}{<} \delta$ then $\varrho_\alpha(f(x), f(z)) \overset{\alpha}{<} \epsilon$. Equivalently, we say that f is *α-continuous* in $x \in \mathbb{R}_\alpha$, if for all $\epsilon \overset{\alpha}{>} \alpha(0)$ there exists a non-Newtonian real number $\delta \overset{\alpha}{>} \alpha(0)$ such that $f(B_\alpha(x, \delta)) \subset B_\alpha(f(x), \epsilon)$. Additionally, we can verify that f is α-continuous in $x \in \mathbb{R}_\alpha$ if, and only if,

$$ {}^{\alpha}\lim_{h\to\alpha(0)} f(x \overset{\alpha}{\oplus} h) = f(x). $$

We say that the function f is α-continuous over $[a, b]_\alpha$ if f is α-continuous for all $x \in [a, b]_\alpha$. We denote by $\mathcal{C}_\alpha[a, b]$ the set of functions $f\colon [a, b]_\alpha \to \mathbb{R}_\alpha$ that are α-continuous over $[a, b]_\alpha$.

Analogously, we say that the function f is α-differentiable in $x \in \mathbb{R}_\alpha$ if the following limit exists:

$$ {}^{\alpha}\lim_{h\to\alpha(0)} ((f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) \overset{\alpha}{\oslash} h). $$

In such a case, the α-derivative of f at $x \in \mathbb{R}_\alpha$ is denoted by

$$ \frac{\mathrm{d}_\alpha f(x)}{\mathrm{d}_\alpha x} = {}^{\alpha}\lim_{h\to\alpha(0)} ((f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) \overset{\alpha}{\oslash} h). \tag{20} $$

We say that the function f is α-differentiable over $[a, b]_\alpha$ if f is α-differentiable for all $x \in [a, b]_\alpha$. We denote by $\mathcal{C}^1_\alpha[a, b]$ the set of functions $f\colon [a, b]_\alpha \to \mathbb{R}_\alpha$ that are α-differentiable over $[a, b]_\alpha$ and such that its α-derivative is α-continuous over $[a, b]$.

We note that if f is α-differentiable in $x \in \mathbb{R}_\alpha$, then f is also α-continuous in x, which follows from observe that

$$ \begin{aligned} {}^{\alpha}\lim_{h\to\alpha(0)} (f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) &= {}^{\alpha}\lim_{h\to\alpha(0)} (((f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) \overset{\alpha}{\oslash} h) \overset{\alpha}{\odot} h) \\ &= \left({}^{\alpha}\lim_{h\to\alpha(0)} ((f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) \overset{\alpha}{\oslash} h) \right) \overset{\alpha}{\odot} \left({}^{\alpha}\lim_{h\to\alpha(0)} h \right) \\ &= \frac{\mathrm{d}_\alpha f(x)}{\mathrm{d}_\alpha x} \overset{\alpha}{\odot} \alpha(0) = \alpha(0), \end{aligned} $$

which shows the result.

We consider an arbitrary function $f\colon \mathbb{R}_\alpha \to \mathbb{R}_\alpha$ and let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ be the function such that the following diagram is commutative:

$$ \begin{array}{ccc} \mathbb{R}_\alpha & \xrightarrow{\ f\ } & \mathbb{R}_\alpha \\ {\scriptstyle \alpha^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^{-1}} \\ \mathbb{R} & \xrightarrow{\ \hat{f}\ } & \mathbb{R} \end{array} \tag{21} $$

that is, such that $\hat{f}(\alpha^{-1}(x)) = (\alpha^{-1} \circ f \circ \alpha)(\alpha^{-1}(x))$ is satisfied for each $x \in \mathbb{R}_\alpha$. We note that if $\alpha\colon \mathbb{R} \to \mathbb{R}_\alpha$ is continuous, then from the diagram (21) it follows that

${}^{\alpha}\lim_{x\to a} f(x) = \ell$ if, and only if, $\lim_{\alpha^{-1}(x)\to\alpha^{-1}(a)} \hat{f}(\alpha^{-1}(x)) = \alpha^{-1}(\ell)$. The above means that

$$ {}^{\alpha}\lim_{x\to a} f(x) = \alpha\left(\lim_{\alpha^{-1}(x)\to\alpha^{-1}(a)} \hat{f}(\alpha^{-1}(x))\right). \tag{22}$$

Furthermore, if $\alpha\colon \mathbb{R}\to\mathbb{R}$ is continuous and $f\colon \mathbb{R}_\alpha \to \mathbb{R}_\alpha$ is α-differentiable in $x\in\mathbb{R}_\alpha$, then from diagram (21) it follows once again that:

$$\begin{aligned} \frac{\mathrm{d}_\alpha f(x)}{\mathrm{d}_\alpha x} &= {}^{\alpha}\lim_{h\to\alpha(0)} ((f(x \overset{\alpha}{\oplus} h) \overset{\alpha}{\ominus} f(x)) \overset{\alpha}{\oslash} h) \\ &= \lim_{\alpha^{-1}(h)\to 0} \alpha\left(\frac{\hat{f}(\alpha^{-1}(x)+\alpha^{-1}(h)) - \hat{f}(\alpha^{-1}(x))}{\alpha^{-1}(h)}\right) = \alpha\left(\frac{\mathrm{d}\hat{f}(\alpha^{-1}(x))}{\mathrm{d}\alpha^{-1}(x)}\right), \end{aligned} \tag{23}$$

whenever $\hat{f}$ is differentiable in $\alpha^{-1}(x)$. Other properties analogous to (22) and (23) can be consulted in [21].

The properties that have been described show that there is a dependence between Newtonian differential calculus and non-Newtonian differential calculus, which follows from the dependence that exists between the topology of the space of Newtonian real numbers and the topology of the space of non-Newtonian real numbers.

There is a similar dependence between Newtonian integral calculus and non-Newtonian integral calculus. We briefly mention this dependence. Let $f\colon \mathbb{R}_\alpha \to \mathbb{R}_\alpha$. The non-Newtonian *$\alpha$-integral*, denoted by ${}^{\alpha}\int_a^x f(\eta)\,\mathrm{d}\eta$, is defined by the requirement that, under typical assumptions parallel to those of the fundamental theorem of Newtonian calculus, it must be satisfied

$$\frac{\mathrm{d}_\alpha}{\mathrm{d}_\alpha x}\, {}^{\alpha}\!\!\int_a^x f(y)\,\mathrm{d}y = f(x), \qquad {}^{\alpha}\!\!\int_a^x \frac{\mathrm{d}_\alpha f(x)}{\mathrm{d}_\alpha x}\,\mathrm{d}x = f(b) \overset{\alpha}{\ominus} f(a),$$

which uniquely implies that

$$ {}^{\alpha}\!\!\int_a^x f(x)\,\mathrm{d}x = \alpha\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \hat{f}(\alpha^{-1}(x))\,\mathrm{d}\alpha^{-1}(x)\right),$$

where $\hat{f}$ is the function obtained from the commutative diagram (21); see [21].

Now we propose to verify that there is also a dependence in the set of Newtonian differential equations and non-Newtonian differential equations, as a particular case, we verify that this property is conserved in the set of first-order linear differential equations.

4 Reachability tube of the non-Newtonian first-order linear differential equations

Using the α-derivative, we can consider finding the solutions of the non-Newtonian first-order linear differential equation with an external perturbation

$$\frac{\mathrm{d}_\alpha y(x)}{\mathrm{d}_\alpha x} \overset{\alpha}{\oplus} \mu \overset{\alpha}{\odot} y(x) = u(x), \quad y(\alpha(0)) = \alpha(0), \tag{24}$$

where μ is a non-Newtonian real number and $u(x)$ is an external perturbation that belongs to the set of functions

$$\mathcal{U}_\delta = \{u(x) \in \mathcal{C}_\alpha[\alpha(0), b] \mid \delta_- \overset{\alpha}{\leq} u(x) \overset{\alpha}{\leq} \delta_+\}, \tag{25}$$

where $b \overset{\alpha}{>} \alpha(0)$ is fixed, δ_- and δ_+ are positive non-Newtonian real numbers that satisfy the inequality $\delta_- \overset{\alpha}{<} \alpha(0) \overset{\alpha}{<} \delta_+$.

We note that (24)–(25) describes a family of non-Newtonian first-order linear differential equations whose elements are indexed by an inclusion $u(x) \in \mathcal{U}_\delta$. In this family we consider the following set:

$$\mathcal{Q}_\alpha = \{(x, y(x))^\top \in \mathbb{R}^2_\alpha \mid y(x) \text{ satisfies (24) for some } u(x) \in \mathcal{U}_\delta \text{ with } x \in [\alpha(0), b]_\alpha\},$$

which is called the *reachability tube* of the family of non-Newtonian first-order linear differential equations (24)–(25). The importance of the reachability tube $\mathcal{Q}_\alpha$ in control theory is due to the following geometric interpretation: this set contains all trajectories of the solutions of the family of non-Newtonian first-order linear differential equations (27)–(28).

In what follows we propose to determine the set $\mathcal{Q}_\alpha$.

If we consider the functions $\hat{y}$ and $\hat{u}$ that are associated with y and u, and such that the following two diagrams are commutative

$$\begin{array}{ccc} \mathbb{R}_\alpha & \xrightarrow{\;y\;} & \mathbb{R}_\alpha \\ {\scriptstyle \alpha^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha^{-1}} \\ \mathbb{R} & \xrightarrow{\;\hat{y}\;} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \mathbb{R}_\alpha & \xrightarrow{\;u\;} & \mathbb{R}_\alpha \\ {\scriptstyle \alpha^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha^{-1}} \\ \mathbb{R} & \xrightarrow{\;\hat{u}\;} & \mathbb{R} \end{array} \tag{26}$$

that is, such that $\hat{y}(\alpha^{-1}(x)) = (\alpha^{-1} \circ y \circ \alpha)(\alpha^{-1}(x))$ and $\hat{u}(\alpha^{-1}(x)) = (\alpha^{-1} \circ u \circ \alpha)(\alpha^{-1}(x))$ for each $x \in \mathbb{R}_\alpha$, and if we use the relation (23), then we can rewrite the non-Newtonian first-order differential equation (23) as

$$\frac{\mathrm{d}\hat{y}(\xi)}{\mathrm{d}\xi} + \alpha^{-1}(\mu)\hat{y}(\xi) = \hat{u}(\xi), \quad \hat{y}(0) = 0, \tag{27}$$

where $\xi = \alpha^{-1}(x)$ for each $x \in \mathbb{R}_\alpha$. On the other hand, we observe that if $u(x) \in \mathcal{U}_\delta$, then $\hat{u}(\xi) \in \hat{\mathcal{U}}$, where

$$\hat{\mathcal{U}} = \{\hat{u}(\xi) \in \mathcal{C}[0, \alpha^{-1}(b)] \mid \alpha^{-1}(\delta_-) \leq \hat{u}(\xi) \leq \alpha^{-1}(\delta_+)\}. \tag{28}$$

We observe that the resulting family of Newtonian first-order linear differential equations (27)–(28) is equivalent to the family of non-Newtonian first-order linear differential equations (24)–(25) under the action of the α-generator. Therefore, we can formulate the problem of determining the reachability tube for the resulting family of Newtonian first-order linear differential equations (27)–(28), which is described by

$$\hat{\mathcal{Q}} = \{(\xi, \hat{y}(\xi))^\top \in \mathbb{R}^2 \mid \hat{y}(\xi) \text{ satisfies (27) for some } \hat{u}(\xi) \in \hat{\mathcal{U}}_\delta \text{ with } \xi \in [0, \alpha^{-1}(b)]\},$$

This reachability tube is well known and has been used to establish a robust stability criterion for a family of Newtonian first-order linear differential equations similar to

(27)–(28), and which have been used to establish a robust stability criterion in some cases of the heat equation; see [25].

We list some of its main properties.

We first observe that if we choose an arbitrary external perturbation $\hat{u}(\xi) \in \hat{\mathcal{U}}_\delta$, and if we substitute this into (27), then we obtain the solution

$$\hat{y}(\xi) = \int_0^\xi e^{-\alpha^{-1}(\mu)\eta}\hat{u}(\xi - \eta)\,\mathrm{d}\eta, \quad \xi \in [0, \alpha^{-1}(b)].$$

On the other hand, if we choose constant external perturbations $\hat{u}^\pm(\xi) = \alpha^{-1}(\delta_\pm)$, and if we substitute these into (27), then the corresponding solutions are

$$\hat{y}^\pm(\xi) = \alpha^{-1}(\delta_\pm)\int_0^\xi e^{-\alpha^{-1}(\mu)\eta}\,\mathrm{d}\eta = \frac{\alpha^{-1}(\delta_\pm)}{\alpha^{-1}(\mu)}\left(1 - e^{-\alpha^{-1}(\mu)\xi}\right), \quad \xi \in [0, \alpha^{-1}(b)].$$

We observe that the functions $\hat{y}^\pm(\xi)$ describe the boundary of the set $\hat{\mathcal{Q}}$, since if $\hat{u}(\xi) \in \hat{\mathcal{U}}_\delta$ is any arbitrary perturbation, then the following inequality is satisfied

$$\hat{u}^-(\eta) \leq \hat{u}(\eta) \leq \hat{u}^-(\eta), \quad \eta \in [0, \alpha^{-1}(b)],$$

Therefore, if we multiply the members of this inequality by $e^{-\alpha^{-1}(\mu)\eta}$, and if we then integrate them from 0 to ξ, then we obtain that

$$\hat{y}^-(\xi) \leq \hat{y}(\xi) \leq \hat{y}^+(\xi), \quad \xi \in [0, \alpha^{-1}(b)].$$

As a consequence of this property, the external perturbations $\hat{u}^\pm(\xi)$ are called *Newtonian worst external perturbations.* We also observe that the set of trajectories obtained from the family of first-order linear differential equations (27)-(28) fill the set $\hat{\mathcal{Q}}$, since if (ξ_0, ν_0) is a point such that $\xi_0 \in [0, \alpha^{-1}(b)]$ and $\hat{y}^-(\xi_0) \leq \nu_0 \leq \hat{y}^+(\xi_0)$, then there exists $\lambda \in [0, 1]$ such that $\nu_0 = \lambda\hat{y}^-(\xi_0) + (1 - \lambda)\hat{y}^+(\xi_0)$. In this case, if we choose the external perturbation $\hat{u}^*(\xi) = \lambda\hat{u}^-(\xi) + (1 - \lambda)\hat{u}^+(\xi) \in \hat{\mathcal{U}}_\delta$, and we substitute it in (27), then the corresponding solution is described by $\hat{y}^*(\xi) = \lambda\hat{y}^-(\xi) + (1 - \lambda)\hat{y}^+(\xi)$, which satisfies $\hat{y}^*(\xi_0) = \nu_0$. This shows that (ξ_0, η_0) is a point that belongs to $\hat{\mathcal{Q}}$.

The above description shows that

$$\hat{\mathcal{Q}} = \{(\xi, \lambda\hat{y}^-(\xi) + (1 - \lambda)\hat{y}^+(\xi))^\top \in \mathbb{R}^2 \mid \text{with } \lambda \in [0, 1] \text{ and } \xi \in [0, \alpha^{-1}(b)]\}.$$

Furthermore, we observe that

$$\hat{\mathcal{Q}} \subset \bar{\mathbb{R}}^+ \times \left[\frac{\alpha^{-1}(\delta_-)}{\alpha^{-1}(\mu)}, \frac{\alpha^{-1}(\delta_+)}{\alpha^{-1}(\mu)}\right],$$

since $\lim_{\xi\to+\infty}\hat{y}^\pm(\xi) = \frac{\alpha^{-1}(\delta_\pm)}{\alpha^{-1}(\mu)}$.

On the other hand, from the first diagram of (26) we observe that if $y(x)$ is a solution associated with an external perturbation $u(x) \in \mathcal{U}_\delta$, then $y(x) = \alpha(\hat{y}(\alpha^{-1}(x)))$ for each $x \in [\alpha(0), b]$, that is

$$y(x) = \alpha\left(\int_0^{\alpha^{-1}(x)} e^{-\alpha^{-1}(\mu)\eta}\hat{u}(\alpha^{-1}(x) - \eta)\,\mathrm{d}\eta\right), \quad x \in [\alpha(0), b].$$

Furthermore, if α is a continuous function, then we obtain the equality $\mathcal{Q}_\alpha = \alpha(\hat{\mathcal{Q}})$ and, as a particular case, we note that the boundary of the reachability tube $\mathcal{Q}_\alpha$ is described by the functions $y^{\pm}(x) = \alpha(\hat{y}^{\pm}(\alpha^{-1}(x)))$, where

$$y^{\pm}(x) = \alpha\left(\frac{\alpha^{-1}(\delta_{\pm})}{\alpha^{-1}(\mu)}\left(1 - e^{-\alpha^{-1}(\mu)\alpha^{-1}(x)}\right)\right).$$

Equivalently,

$$y^{\pm}(x) = (\delta_{\pm} \overset{\alpha}{\oslash} \mu) \overset{\alpha}{\odot} \alpha(1 - e^{-\alpha^{-1}(\mu)\alpha^{-1}(x)}). \tag{29}$$

The solutions in (29) are obtained as an effect of the external perturbations $u^{\pm}(x) = \delta_{\pm}$ for each $x \in [\alpha(0), b]_\alpha$. We also note that $u^{\pm}(x) = \alpha(\hat{u}^{\pm}(\alpha^{-1}(x)))$ for each $x \in [\alpha(0), b]_\alpha$. Furthermore, due to the order of the non-Newtonian real numbers, it follows that if $u(x) \in \mathcal{U}_\delta$ is an arbitrary external perturbation, then the following inequalities are satisfied

$$y^{-}(x) \overset{\alpha}{\leq} y(x) \overset{\alpha}{\leq} y^{+}(x), \qquad x \in [\alpha(0), b].$$

We note that as a consequence of this property, the external perturbations $u^{\pm}(x)$ will be called *worst non-Newtonian external perturbations*. On the other hand, as a consequence of the previous inequalities, we can observe that

$$\hat{\mathcal{Q}} = \{(x, \lambda \overset{\alpha}{\odot} y^{-}(x) \overset{\alpha}{\oplus} (\alpha(1) \overset{\alpha}{\ominus} \lambda) \overset{\alpha}{\odot} y^{+}(\xi))^{\top} \in \mathbb{R}^2_\alpha \mid \text{with } \lambda \in [\alpha(0), \alpha(1)]_\alpha \text{ and } x \in [\alpha(0), b]\}.$$

Finally, we observe that $\mathcal{Q}_\alpha \subset \bar{\mathbb{R}}^{+}_\alpha \times \left[\delta_{-} \overset{\alpha}{\oslash} \mu, \delta_{+} \overset{\alpha}{\oslash} \mu\right]_\alpha$, since ${}^{\alpha}\lim_{x\to\alpha(+\infty)} y^{\pm}(x) = \delta_{\pm} \overset{\alpha}{\oslash} \mu$, where $\alpha(+\infty) = \lim_{\xi\to+\infty} \alpha(\xi)$.

We summarize the results obtained as follows.

Theorem 4.1. *Consider the non-Newtonian first-order linear differential equation*

$$\frac{\mathrm{d}_\alpha y(x)}{\mathrm{d}_\alpha x} \overset{\alpha}{\oplus} \mu \overset{\alpha}{\odot} y(x) = u(x), \quad y(\alpha(0)) = \alpha(0),$$

where μ is a non-Newtonian real number and $u(x) \in \mathcal{U}_\delta$ is an external perturbation. Then the solution to this non-Newtonian first-order linear differential equation is

$$y(x) = \alpha\left(\int_0^{\alpha^{-1}(x)} e^{-\alpha^{-1}(\mu)\eta}\hat{u}(\alpha^{-1}(x) - \eta)\,\mathrm{d}\eta\right), \quad x \in [\alpha(0), b].$$

On the other hand, the boundary of the reachability tube $\mathcal{Q}_\alpha$ is represented by the graph of the functions

$$y^{\pm}(x) = \alpha\left(\frac{\alpha^{-1}(\delta_{\pm})}{\alpha^{-1}(\mu)}\left(1 - e^{-\alpha^{-1}(\mu)\alpha^{-1}(x)}\right)\right), \quad x \in [\alpha(0), b].$$

An analogous result can be obtained if we consider an initial condition $y(\alpha(0)) \neq \alpha(0)$.

The following example shows the reachability tube obtained in a non-Newtonian first-order linear differential equation using some particular cases of non-Newtonian calculus.

Example 4.2. We choose $\mu = \alpha(2)$ and $\delta_\pm = \alpha(\pm 1)$, and consider as a particular case the following non-Newtonian first-order linear differential equation defined on $[\alpha(0), \alpha(2)]_\alpha$:

$$\frac{\mathrm{d}_\alpha y(x)}{\mathrm{d}_\alpha x} \overset{\alpha}{\oplus} \alpha(2) \overset{\alpha}{\odot} y(x) = u(x), \quad y(\alpha(0)) = \alpha(0),$$

with $u(x) \in \mathcal{U}_\delta$, where $\mathcal{U}_\delta = \{u(x) \in \mathcal{C}_\alpha[\alpha(0), \alpha(2)] \mid \delta_- \overset{\alpha}{\leq} u(x) \overset{\alpha}{\leq} \delta_+\}$. We obtain the following three particular cases of the functions $y^\pm(x)$ that describe the boundary of the reachability tube $\mathcal{Q}_\alpha$ when $x \in [\alpha(0), \alpha(2)]$. We note that all three graphs are plotted in $\mathbb{R}^2$ to compare the structure of the reachability tube $\mathcal{Q}_\alpha$ in each case.

If we choose $\alpha(x) = id(x)$, that is, if we consider a Newtonian calculus, then the boundary of the reachability tube $\mathcal{Q}_\alpha$ is described by the functions

$$y^\pm(x) = \pm(1 - e^{-2x}), \quad x \in [0, 2]. \tag{30}$$

An illustration of the corresponding reachability tube is shown in Figure 1.

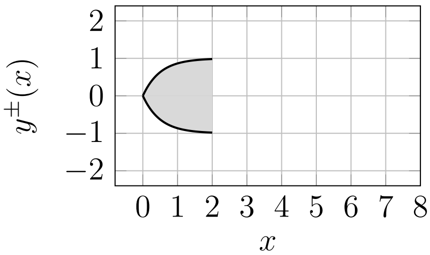

Fig. 1. Reachability tube in a Newtonian calculus with boundary described by the functions (30).

On the other hand, if we assume that $\alpha(x) = \exp(x)$, that is, if we consider a multiplicative calculus, or geometric calculus, then the boundary of the reachability tube $\mathcal{Q}_\alpha$ is described by functions

$$y^\pm(x) = e^{\pm\frac{1}{2}\left(1 - \frac{1}{x^2}\right)}, \quad x \in [1, e^2]_{\exp}. \tag{31}$$

An illustration of the corresponding reachability tube is shown in Figure 2.

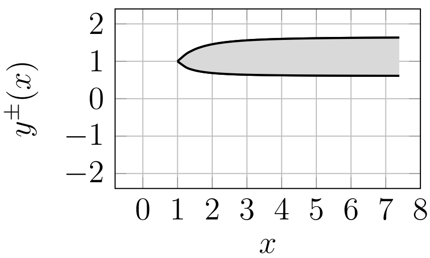

Fig. 2. Reachability tube in a multiplicative calculus, or geometric calculus, with boundary described by the functions (31).

Finally, if we consider that $\alpha(x) = \kappa^{-1}\sinh(\kappa x)$, that is, if we consider a Kaniadakis κ-calculus, then the boundary of the reachability tube $\mathcal{Q}_\alpha$ is described by the functions

$$y^\pm(x) = \pm\kappa^{-1}\sinh\left(\tfrac{1}{2}\kappa\left(1 - e^{-2\kappa\sinh(\kappa x)}\right)\right), \quad x \in \left[0, \kappa^{-1}\sinh(2\kappa)\right]_\kappa. \tag{32}$$

An illustration of the corresponding reachability tube is shown in Figure 3 with $\kappa = \frac{9}{10}$.

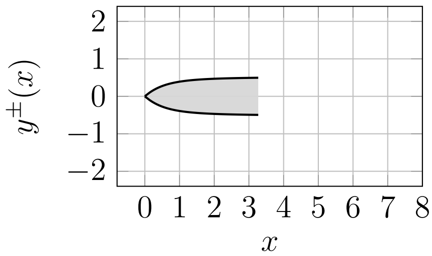

Fig. 3. Reachability tube in a Kaniadakis κ-calculus with boundary described by the functions (32).

Other values for the parameters and other α-generators can be chosen to obtain similar illustrations of the corresponding reachability tube $\mathcal{Q}_\alpha$.

5 Conclusion

This paper presented the algebraic and topological properties of non-Newtonian real numbers, as well as some elements of non-Newtonian differential calculus. Using the definition of a non-Newtonian derivative, a non-Newtonian first-order linear differential equation that admits external perturbations has been considered, and for this, the problem of determining the reachability tube has been posed.

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