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A basic epistemic logic and its algebraic model

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Abstract: In this paper we propose an algebraic model for a modal epistemic logic. Although it is known the existence of algebraic models for modal logics, considering that there are so many different modal logics, so it is not usual to give an algebraic model for each such system. The basic epistemic logic used in the paper is bimodal and we can show that the epistemic algebra introduced in the paper is an adequate model for it.

Keywords: Epistemic logic; Knowledge and belief; Algebraic logic; Algebraic model.

Classification MSC: 03B45; 03G27; 03C60.

Resumo: Neste artigo propomos um modelo algébrico para uma bem simples lógica modal epistêmica. Embora seja conhecida a existência de modelos algébricos para as lógicas modais, devido ao fato de que existe uma quantidade enorme de lógicas modais, não é usual apresentarmos modelos algébricos para cada tal particular caso modal. O sistema de lógica epistêmica usado nestas notas é bimodal e podemos demonstrar que a álgebra espistêmica introduzida no artigo é modelo algébrico adequado à lógica considerada.

Palavras-chave: Lógica epistêmica; Conhecimento e crença; Lógica algébrica; Modelo algébrico.

1 Introduction

Algebraic logic formalizes some aspects of logic and then places these aspects in a general algebraic environment. Usually, the constructions with algebraic logic consider and develop some other branches of mathematics, such as topology, filters and ideals, and set theory, among others.

The contemporary logics have several types of models, and algebraic models are one of these distinct contexts for logical interpretation.

The first steps of algebraic logic appeared in the XIX century with Boole and other thinkers, but they were well-developed in the Polish tradition with Tarski, Rasiowa and Sikorski, in the next century (Dunn and Hardgree [4]), (Rasiowa and Sikorski [10]).

In general, it is easier to recognize the properties of a logic in its models than in its deductive systems. In particular, the algebraic models can express many logical laws in a simple way.

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Of course, it is more direct for people with some mathematical experience, but in the next pages we will observe that it is not difficult to see an algebraic model for a very basic logic.

For the purposes of this article, we selected an Epistemic Logic (EL) among the formal systems proposed to model this relationship between known facts and the facts about knowing highlights. One can find more information about the relationship between knowledge and Epistemic Logic in (Meyer and Van der Hoek [7]).

In 1999, Mortari [9] points out that the notion of knowledge is linked to the verb *to know*, a verb that is said of propositional attitudes, i.e, *knowledge* refers to "attitudes that an intelligent agent can have in relation to any proposition p".

Thus, the formalization of epistemic logics can connect aspects of epistemology with the most up-to-date technologies of information and I.A. devices (Halpern [5]), (Rosenschein [12]).

In Section 2, we present the modal system of Basic Epistemic Logic **BEL**. In the next section, we introduce, as an original contribution, the correspondent algebra $\mathcal{B}$, for short BEL-algebra, motivated by the algebraic logic tradition and interest in **BEL**. In Sections 4 and 5 we show the theorems of soundness and completeness, developed for the class of BEL-algebras, which are shown to be completely adequate to the logical system **BEL**.

2 A basic epistemic logic

In this section we present the system of epistemic logic for which we will give an algebraic model. We follow Mortari [9] and denote this logic by **BEL**.

Modal logics have been deeply studied in (Carnielli and Pizzi [1]), (Chagrov and Zakharyachev [2]) and (Chellas [3]).

The propositional logic **BEL** is constructed over the propositional language $L = \{\neg, \wedge, \to, K_i, B_i, p_1, p_2, p_3, ...\}$, where *knowledge* and *belief* are represented via the modal operators K_i and B_i, for some $m, 1 \leq i \leq m$. Also $\top$ ("top") and $\bot$ ("bottom") are used to denote the constantly true proposition and the constantly false proposition, respectively. The calculus for the system is the Hilbert calculus for classical propositional logic with the following additional axioms and rules:

(CPC) φ, if φ is a tautology

(K_K) $K(\varphi \to \psi) \to (K\varphi \to K\psi)$

(K_B) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$

(T) $K\varphi \to \varphi$

(D) $B\varphi \to \neg B\neg\varphi$

(M) $K\varphi \to B\varphi$

(MP) $\varphi \to \psi, \quad \varphi \;/\; \psi$

(RN_K) $\vdash \varphi \;/\; \vdash K\varphi.$

Formulas $K_a\varphi$ and $B_a\varphi$ are then read like "agent a knows that φ" and "agent a believes that φ". We will not detach the agents $a, b, \ldots$.

As usual, we write $\vdash \varphi$ to indicate that φ is a theorem.

So it holds $K(\varphi \vee \neg\varphi)$ or $K\ \top$, considering that $\top$ denotes a random theorem of **BEL**.

If $\Gamma \cup \{\varphi\}$ is a set of formulas, then Γ deduces φ, what is denoted by $\Gamma \vdash \varphi$, if there is a finite sequence of formulas $\varphi_1, ..., \varphi_n$ such that $\varphi_n = \varphi$ and, for every φ_i, $1 \leq i \leq n$:

- φ_i is an axiom, or
- $\varphi_i \in \Gamma$, or
- φ_i is obtained from previous formulas of the sequence by some of the deduction rules.

Let For_{BEL} be the set of all propositional formulas and Var_{BEL} be the set of all propositional variables of the system **BEL**.

We don't need to add RN_B as an inference rule because it can be derived from (RN_K) and (M), see below:

Proposition 2.1. *(i)* [RN_B] If $\vdash \varphi$, then $\vdash B\varphi$;

(ii) If $\vdash \varphi \to \psi$, then $\vdash K\varphi \to K\psi$;

(iii) If $\vdash \varphi \to \psi$, then $\vdash B\varphi \to B\psi$.

Proof: (i)

1. φ	Hypothesis
2. $K\varphi \to B\varphi$	M
3. $K\varphi$	RN_K in 1
4. $B\varphi$	MP in 2 and 3.

(ii)

1. $\varphi \to \psi$	Hypothesis
2. $K(\varphi \to \psi)$	RN_K in 1
3. $K(\varphi \to \psi) \to (K\varphi \to K\psi)$	K_K
4. $K\varphi \to K\psi$	MP in 2 and 3.

(iii) As (ii) using RN_B. ■

Proposition 2.2. The following formulas are theorems:

(i) $(B\varphi \wedge B\psi) \leftrightarrow B(\varphi \wedge \psi)$;

(ii) $(K\varphi \wedge K\psi) \leftrightarrow K(\varphi \wedge \psi)$;

(iii) $\neg K\bot$.

Proof:

(i) ($\Rightarrow$)

1. $\varphi \to (\psi \to (\varphi \wedge \psi))$ CPC
2. $B(\varphi \to (\psi \to (\varphi \wedge \psi)))$ RN_B in 1
3. $B(\varphi \to (\psi \to (\varphi \wedge \psi))) \to (B\varphi \to B(\psi \to (\varphi \wedge \psi)))$ K_B
4. $B\varphi \to B(\psi \to (\varphi \wedge \psi))$ MP in 2 and 3
5. $B(\psi \to (\varphi \wedge \psi)) \to (B\psi \to B(\varphi \wedge \psi))$ K_B
6. $B\varphi \to (B\psi \to B(\varphi \wedge \psi))$ CPC in 4 and 5
7. $(B\varphi \wedge B\psi) \to B(\varphi \wedge \psi)$ CPC in 6.

$(\Leftarrow)$

1. $(\varphi \wedge \psi) \to \varphi$ CPC
2. $B(\varphi \wedge \psi) \to B\varphi$ 2.1 (iii) in 1
3. $(\varphi \wedge \psi) \to \psi$ CPC
4. $B(\varphi \wedge \psi) \to B\psi$ 2.1 (iii) in 3
5. $B(\varphi \wedge \psi) \to (B\varphi \wedge B\psi)$ CPC in 2 and 4.

(ii) The proof is similar to the previous item using the axiom K_K and the rule RN_K.

(iii)

1. $K\bot \to \bot$ T
2. $\neg\bot \to \neg K\bot$ Contrapositive in 1.
3. $\top \to \neg K\bot$ Substitution in 3. ■
4. $\top$ Theorem
5. $\neg K\bot$ MP in 3 and 4.

3 The epistemic algebra

Considering the system **BEL**, we introduce the correspondent algebra $\mathcal{B}$, BEL-algebra for short.

More information about algebraic models for logical systems can be seen in (Dunn and Hardgree [4]), (Miraglia [8]), (Rasiowa and Sikorski [10]) and (Rasiowa [11]).

Definition 3.1. An epistemic algebra is a tuple $\mathcal{B} = (B, 0, 1, \sim, \sqcap, \sqcup, \Bbbk, \flat)$ such that $(B, 0, 1, , \sim, \sqcap, \sqcup)$ is a Boolean algebra, $\Bbbk : B \to B$ is an operator for the notion of knowledge, and $\flat : B \to B$ is an operator for the notion of belief, such that:

(a) $\flat 0 = 0$

(b) $\flat(a \sqcap b) = \flat a \sqcap \flat b$

(c) $\Bbbk(a \sqcap b) = \Bbbk a \sqcap \Bbbk b$

(d) $\Bbbk 1 = 1$

(e) $\Bbbk a \leq a$

(f) $\Bbbk a \leq \flat a$.

From items (a), (d) and (f) it is immediate that $\Bbbk 0 = 0$ and $\flat 1 = 1$. We do not have any data on the order for a and $\flat a$.

Considering that $\mathcal{B}$ is Boolean, then $a \to b = \sim a \sqcup b$ and naturally the De Morgan laws are valid.

Proposition 3.2. For any Boolean algebra, it holds the following equivalence:

$a \leq b \Leftrightarrow \; a \to b = 1$.

Proof: If $a \leq b$, then $a \sqcup b = b \Rightarrow \sim a \sqcup (a \sqcup b) = \sim a \sqcup b \Rightarrow (\sim a \sqcup a) \sqcup b = \sim a \sqcup b \Rightarrow 1 \sqcup b = a \to b \Rightarrow 1 = a \to b$.

If $a \to b = 1$, then $\sim a \sqcup b = 1 \Rightarrow a \sqcap (\sim a \sqcup b) = a \sqcap 1 \Rightarrow (a \sqcap \sim a) \sqcup (a \sqcap b) = a \Rightarrow 0 \sqcup (a \sqcap b) = a \Rightarrow a \sqcap b = a \Rightarrow a \leq b$. ■

Proposition 3.3. If $\mathcal{B} = (B, 0, 1, \sim, \sqcap, \sqcup, \Bbbk, \flat)$ is an epistemic algebra, then:

(i) $a \leq b \Rightarrow \Bbbk a \leq \Bbbk b$;

(ii) $a \leq b \Rightarrow \flat a \leq \flat b$.

Proof: (i) If $a \leq b$, then $a \sqcap b = a \Rightarrow \Bbbk(a \sqcap b) = \Bbbk a \Rightarrow \Bbbk a \sqcap \Bbbk b = \Bbbk a \Rightarrow \Bbbk a \leq \Bbbk b$.

(ii) The same justification. ■

Proposition 3.4. If $\mathcal{B} = (B, 0, 1, \sim, \sqcap, \sqcup, \Bbbk, \flat)$ is an epistemic algebra, then:

(i) $a \sqcap z \leq b \Rightarrow z \leq a \to b$;

(ii) $\Bbbk(a \to b) \leq (\Bbbk a \to \Bbbk b)$.

Proof: (i) If $a \sqcap z \leq b$, then $z \to (a \sqcap z) \leq z \to b \Rightarrow a \to (z \to (a \sqcap z)) \leq a \to (z \to b)$. As $a \to (z \to (a \sqcap z)) = 1$, then $a \to (z \to b) = 1$ and $z \to (a \to b) = 1$. Thus $z \leq a \to b$.

(ii) For this item we use the items (c) of Definition 3.1 and (i) of Propositions 3.3 and 3.4.

$a \sqcap b \leq b \Rightarrow 0 \sqcup (a \sqcap b) \leq b \Rightarrow (a \sqcap \sim a) \sqcup (a \sqcap b) \leq b \Rightarrow a \sqcap (\sim a \sqcup b) \leq b \Rightarrow \Bbbk(a \sqcap (\sim a \sqcup b)) \leq \Bbbk b \Rightarrow \Bbbk(a) \sqcap \Bbbk(\sim a \sqcup b)) \leq \Bbbk b \Rightarrow \Bbbk(a) \sqcap \Bbbk(a \to b)) \leq \Bbbk b \Rightarrow \Bbbk(a \to b) \leq (\Bbbk a \to \Bbbk b)$.
■

4 Soundness

Now, we need to show that the class of BEL-algebras is an appropriate model for **BEL**.

Definition 4.1. A restrict valuation is a function $\overline{v} : Var_{BEL} \to \mathcal{B}$, that maps each variable of **BEL** in an element of $\mathcal{B}$.

Definition 4.2. A valuation is a function $v : For_{BEL} \to \mathcal{B}$, that extends natural and uniquely $\overline{v}$ as follows:

(i) $v(p) = \overline{v}(p)$

(ii) $v(\neg\varphi) = \sim v(\varphi)$

(iii) $v(\varphi \wedge \psi) = v(\varphi) \sqcap v(\psi)$

(iv) $v(\varphi \to \psi) = \sim v(\varphi) \sqcup v(\psi)$

(v) $v(K\varphi) = \Bbbk\ v(\varphi)$

(vi) $v(B\varphi) = \flat\ v(\varphi)$.

As usual, operator symbols in the left sides represent logical operators and those in right sides represent algebraic operators.

Of course, $v(\varphi \vee \psi) = v(\varphi) \sqcup v(\psi)$.

Definition 4.3. A valuation $v : For_{BEL} \to \mathcal{B}$ is a model for a set $\Gamma \subseteq For_{BEL}$ if $v(\varphi) = 1$, for each formula $\varphi \in \Gamma$.

In particular, a valuation $v : For_{BEL} \to \mathcal{B}$ is a model for a formula $\varphi \in For_{BEL}$ when $v(\varphi) = 1$.

Definition 4.4. A formula $\varphi \in For_{BEL}$ is valid in a BEL-algebra $\mathcal{B}$ if each valuation $v : For_{BEL} \to \mathcal{B}$ is a model for φ.

Definition 4.5. A formula φ **BEL**-valid, what is denoted by $\vDash \varphi$, when it is valid in every BEL-algebra.

The soundness theorem must show that the BEL-algebras are correct models for the logic **BEL**, that is, that every theorem of **BEL** is valid in any BEL-algebra and that the **BEL**-rules preserve the validity.

Since each BEL-algebra is a Boolean algebra, then we will use the Proposition 3.2 several times.

Theorem 4.6. (Weak Soundness) If $\vdash \gamma$ then $\vDash \gamma$.
Proof: If $\mathcal{B} = (B, 0, 1, \sim, \sqcap, \sqcup, \Bbbk, \flat)$ is a generic BEL-algebra, then we must show that the axioms $(K_K), (K_B), (T), (D)$ and (M) are valid in $\mathcal{B}$, and that the rule (RN_K), according Definition 3.1, preserves validity in $\mathcal{B}$. The Boolean part works as usually.

(T): By (e), $\Bbbk\ v(\varphi) \leq v(\varphi)$, so considering Proposition 2.2, $\Bbbk\ v(\varphi) \to v(\varphi) = 1$ and then $v(K\varphi \to \varphi) = 1$.

(M): By (f), $\Bbbk\ v(\varphi) \leq \flat\ v(\varphi)$, then $\Bbbk\ v(\varphi) \to \flat\ v(\varphi) = 1$ and $v(K\varphi \to B\varphi) = 1$.

(D): $v(B\varphi \to \neg B\neg\varphi) = \sim (\flat\ v(\varphi) \sqcap \flat\ \sim v(\varphi))$. But by (b) and (a), $\flat\ v(\varphi) \sqcap \flat\ \sim v(\varphi) = \flat\ (v(\varphi) \sqcap \sim v(\varphi)) = \flat 0 = 0$. So, $v(B\varphi \to \neg B\neg\varphi) = 1$.

(K_K): By Proposition 3.4 (ii), we have $v(K(\varphi \to \psi)) \leq v(K\varphi \to K\psi)$ and hence $v(K(\varphi \to \psi) \to (K\varphi \to K\psi)) = 1$.

(K_B): The proof is analogous to (K_K) by using Proposition 3.4.

(RN_K): If $v(\varphi) = 1$, using (d) we have $v(K\varphi) = \Bbbk\ v(\varphi) = \Bbbk 1 = 1$. ■

Corollary 4.7. (Strong Soundness) If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.
Proof: Suppose $\Gamma \vdash \varphi$, and let $\mathcal{B}$ be an algebraic model such that $\mathcal{B} \vDash \Gamma$.

The proof is by induction on the the length of the deduction $\Gamma \vdash \varphi$.

- If $n = 1$, then φ is an axiom (theorem) or belongs to Γ.
  If it is an axiom, the result is given by the preceding theorem. If φ belongs to Γ, then naturally $\Gamma \vDash \varphi$.
- Let now $n > 1$, then φ is obtained by (MP) or (RN_K).
  But these two rules preserve the validity, then $\mathcal{B} \vDash \varphi$.

■

The next corollary shows that if a set of formulas has a model, then it is consistent.

Corollary 4.8. The logic **BEL** is consistent.
Proof: Suppose that **BEL** is not consistent. Then there is $\varphi \in For_{BEL}$ such that $\vdash \varphi$ and $\vdash \neg\varphi$.

So, by the soundness theorem, φ and $\neg\varphi$ are valid. Let v be a valuation in a BEL-algebra with exactly two elements $2 = \{0, 1\}$. Since φ is valid, then $v(\varphi) = 1$ and $v(\neg\varphi) = \sim v(\varphi) = 0$. But this contradicts the fact of $\neg\varphi$ is valid. ■

5 Completeness

For the completeness we will use the Lindenbaum algebras.

Let $(For_{BEL}, \bot, \top, \neg, \wedge, \rightarrow, K, B)$ be the algebra of formulas of **BEL**, such that $\bot$ and $\top$ are constants, $\neg$, K and B are unary operators, $\wedge$ and $\rightarrow$ are binary operators, and as usual $\varphi \vee \psi =_{df} \neg(\neg\varphi \wedge \neg\psi)$.

So, we define the Lindenbaum Algebra of **BEL**.

Definition 5.1. For $\Gamma \subseteq For_{BEL}$, we define the relation $\equiv$ by:

$$\varphi \equiv \psi \Leftrightarrow \ \Gamma \vdash \varphi \rightarrow \psi \ \mathit{and} \ \Gamma \vdash \psi \rightarrow \varphi.$$

The relation $\equiv$, more than an equivalence relation, is a congruence, for by rule RN_K:

$$\varphi \equiv \psi \Leftrightarrow \ \Gamma \vdash \varphi \leftrightarrow \psi \Rightarrow \ \Gamma \vdash K\varphi \leftrightarrow K\psi \Leftrightarrow K\varphi \equiv K\psi.$$

Also, as we have $\vdash \varphi \ \ / \ \ \vdash B\varphi$, then:

$$\varphi \equiv \psi \Leftrightarrow \ \Gamma \vdash \varphi \leftrightarrow \psi \Rightarrow \ \Gamma \vdash B\varphi \leftrightarrow B\psi \Leftrightarrow B\varphi \equiv B\psi.$$

If $\Gamma \cup \{\psi\} \subseteq For_{BEL}$, we denote the equivalence class of ψ modulo $\equiv$ and Γ by: $[\psi]_\Gamma = \{\sigma \in For_{BEL} : \sigma \equiv \psi\}$.

Definition 5.2. The Lindenbaum algebra of **BEL**, denoted by $\mathcal{B}_\Gamma(\mathbf{BEL})$, is the quotient algebra defined by:

$\mathcal{B}_\Gamma(\mathbf{BEL}) = (For_{BEL}|_\equiv, \mathbf{0}, \mathbf{1}, \neg_\equiv, \wedge_\equiv, \rightarrow_\equiv, K_\equiv, B_\equiv)$, such that:

(i) $\mathbf{0} = [\varphi \wedge \neg\varphi] = [\bot]$

(ii) $\mathbf{1} = [\varphi \vee \neg\varphi] = [\top]$

(iii) $\neg_\equiv[\varphi] = [\neg\varphi]$

(iv) $[\varphi] \wedge_\equiv [\psi] = [\varphi \wedge \psi]$

(v) $[\varphi] \rightarrow_\equiv [\psi] = [\varphi \rightarrow \psi]$

(vi) $K_\equiv[\varphi] = [K\varphi]$

(vii) $B_\equiv[\varphi] = [B\varphi]$.

In general, we will not indicate the index $\equiv$ in the operations.

When $\Gamma = \emptyset$ we denote the Lindenbaum algebra of **BEL** by $\mathcal{B}(\mathbf{BEL})$.

Proposition 5.3. In $\mathcal{B}_\Gamma(\mathbf{BEL})$ it holds: $[\varphi] \leq [\psi] \Leftrightarrow \ \Gamma \vdash \varphi \rightarrow \psi$.
Proof: $[\varphi] \leq [\psi] \Leftrightarrow \ [\varphi] \wedge [\psi] = [\varphi] \Leftrightarrow [\varphi \wedge \psi] = [\varphi] \Leftrightarrow \ \Gamma \vdash \varphi \wedge \psi \leftrightarrow \varphi \Leftrightarrow \ \Gamma \vdash \varphi \rightarrow \psi$. ■

Proposition 5.4. The algebra $\mathcal{B}_\Gamma(\mathbf{BEL})$ is a BEL-algebra.
Proof: (a) From axiom D, $B\varphi \rightarrow \neg B\neg\varphi$, we have $\neg(B\varphi \wedge B\neg\varphi)$ iff $\neg(B(\varphi \wedge \neg\varphi))$. Then $[\neg(B(\varphi \wedge \neg\varphi))] = 1$ and $[B(\varphi \wedge \neg\varphi)] = 0$. Then, $[B\bot] = 0 \Rightarrow B[\bot] = 0$.

(b) From Proposition 2.2 (ii), $B(\varphi \wedge \psi) \leftrightarrow B\varphi \wedge B\psi$. Then $[B(\varphi \wedge \psi)] = [B\varphi \wedge B\psi]$ and $[B(\varphi \wedge \psi)] = [B\varphi] \wedge [B\psi]$.

(c) From Proposition 2.2 (i), $K(\varphi \wedge \psi) \leftrightarrow K\varphi \wedge K\psi$. Then $[K(\varphi \wedge \psi)] = [K\varphi \wedge K\psi]$ and $[K(\varphi \wedge \psi)] = [K\varphi] \wedge [K\psi]$.

(d) As $\vdash K(\varphi \vee \neg\varphi)$, then $[K\top] = 1 \Rightarrow K[\top] = 1$.

(e) From (T) $K\varphi \to \varphi$, then $[K\varphi] \leq [\varphi] \Rightarrow K[\varphi] \leq [\varphi]$.

(f) From (M) $K\varphi \to B\varphi$, we have $[K\varphi] \leq [B\varphi] \Rightarrow K[\varphi] \leq B[\varphi]$. ■

Definition 5.5. The algebra $\mathcal{B}_\Gamma(\mathbf{BEL})$ is the canonical model of $\Gamma \subseteq For_{BEL}$.

We denote a valuation on the canonical model by $v_0 : For_{BEL} \to \mathcal{B}_\Gamma(\mathbf{BEL})$. When $\Gamma = \emptyset$ we have $v_0 : For_{BEL} \to \mathcal{B}(\mathbf{BEL})$.

Corollary 5.6. Let $\varphi \in For_{BEL}$ and $\mathcal{B}(\mathbf{BEL})$ be the canonical model for **BEL**. If φ is a theorem of **BEL**, then $[\varphi] = 1$, and if φ is irrefutable, then $[\varphi] \neq 0$.
Proof: If $\vdash \varphi$, considering that $\mathcal{B}(\mathbf{BEL})$ is a BEL-algebra, by the soundness theorem we have $[\varphi] = 1$.

Now, φ is irrefutable iff $\nvdash \neg\varphi$ iff $[\neg\varphi] \neq 1$ iff $\neg[\varphi] \neq 1$ iff $[\varphi] \neq 0$. ■

From the preceding proposition and the definitions of **0** and **1** in the Lindenbaum algebra it results that for each formula φ:

$[\varphi] = 1$ iff $\vdash \varphi$ and
$[\varphi] = 0$ iff $\vdash \neg\varphi$.

Theorem 5.7. For $\varphi \in For_{BEL}$, the following assertions are equivalent:

(i) $\vdash \varphi$;
(ii) $\vDash \varphi$;
(iii) φ is valid in every BEL-algebra of sets $\mathbf{B} = (B, \emptyset, {}^C, \cap, \cup, K, B)$;
(iv) $v_0(\varphi) = \mathbf{1}$, for the canonical valuation in $\mathcal{A}(\mathbf{BEL})$.

Proof: $(i) \Rightarrow (ii)$: from the Soundness Theorem.

$(ii) \Rightarrow (iii)$: is immediate.

$(iii) \Rightarrow (iv)$: as every BEL-algebra is isomorphic to a BEL-algebra of sets $\mathbf{B} = (B, \emptyset, {}^C, \cap, \cup, K, B)$ and $\mathcal{A}(\mathbf{BEL})$ is a BEL-algebra, the result follows.

$(iv) \Rightarrow (i)$: if $\varphi \in For_{BEL}$ and it is not derivable in **BEL**, by Corollary 5.6, $[\varphi] \neq 1$ in $\mathcal{A}(\mathbf{BEL})$ and then $v_0(\varphi) \neq 1$. Therefore φ is not a valid formula. ■

Corollary 5.8. (Completeness) For each $\varphi \in For_{BEL}$, if φ is valid, then φ is derivable in **BEL**.

The next result shows the strong adequacy of the algebraic models given by BEL-algebras for the logic system **BEL**.

As usual, $\Gamma \vDash \varphi$ denotes that every model of Γ is a model of φ too.

Definition 5.9. A model $v : For_{BEL} \to \mathcal{A}$ is strongly adequate for Γ when:

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \vDash \varphi.$$

Proposition 5.10. If $\Gamma \subseteq For_{BEL}$ is consistent, then the canonical valuation is a correct model for Γ.

Proof: Considering the canonical valuation $v_0 : For_{BEL} \to \mathcal{A}(\mathbf{BEL})$, that maps $v_0(\varphi) = [\varphi]$, by Corollary 5.6 and Proposition 4.7, $v_0(\varphi) = 1$ iff $\Gamma \vdash \varphi$. Therefore we have that v_0 is a correct model for Γ. ■

Theorem 5.11. For $\Gamma \subseteq For_{BEL}$, the following conditions are equivalent:

(i) Γ is consistent;

(ii) there is a correct model for Γ;

(iii) there is a correct model for Γ in a BEL-algebra of sets $\mathbf{B} = (B, \emptyset, {}^C, \cap, \cup, K, B)$;

(iv) there is a model for Γ.

Proof: $(i) \Rightarrow (ii)$ As in the previous proposition.

$(ii) \Rightarrow (iii)$ As $\mathcal{A}(\mathbf{BEL})$ is a BEL-algebra and every BEL-algebra is isomorphic to a BEL-algebra of sets $\mathbf{B} = (B, \emptyset, {}^C, \cap, \cup, K, B)$, then the result follows.

$(iii) \Rightarrow (iv)$ Immediate.

$(vi) \Rightarrow (i)$ It results directly by Corollary 4.8. ■

Corollary 5.12. (Strong adequacy) Let $\Gamma \cup \{\varphi\} \subseteq For_{BEL}$. If Γ is consistent, the following conditions are equivalent:

(i) $\Gamma \vdash \varphi$;

(ii) $\Gamma \vDash \varphi$;

(iii) every model of Γ in a BEL-algebra of sets $\mathbf{B} = (B, \emptyset, {}^C, \cap, \cup, K, B)$ is a model for φ;

(iv) $v_0(\varphi) = 1$ for the canonical valuation v_0.

This way, we have shown that the BEL-algebras are adequate models for this basic epistemic logic. If we need some more complex epistemic system to formalize some situation, we think that we can use this algebra as an initial point and get other algebraic models too.

6 Final considerations

Among the several reasons for justifying the aim of any algebraic model for a logic, or an epistemic logic, is the simplicity in the use of algebraic structures, as well as the intuitive use of algebraic notions for problems involved with the present-day technologies (Meyer and Van der Hoek [7]), (Halpern and Moses [6]).

We chose a basic system, but we consider that we can extend the method to several other epistemic systems. Maybe we can think about what more laws can be included into **BEL** and preserve algebraic models.

In another direction, we could study more relations between the two epistemic operators and other operators defined from these.

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