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A note on Repunit number sequence

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Abstract: In this paper, we investigate the classical identities of the repunit sequence with integer indices in light of the properties of Horadan-type sequences. We highlight particularly the Tagiuri-Vajda identity and Gelin-Cesàro identity. Additionally, we prove that no repunit is a perfect power, whether even or odd. Finally, we address a divisibility criterion for the terms of repunit r_n by a prime p and its powers.

Keywords: Horadan-type sequence; Perfect power; Repunit sequence; Tagiuri-Vajda Identity.

Classification MSC: Primary 11B37; Secondary 11B39, 11B83.

1 Introduction

For n natural, consider $\{h_n\}$ the Horadam sequence defined by the recurrence relation of second order, where p and q are fixed integers, such that

$$h_{n+1} = ph_n + qh_{n-1}, \text{for all } n \geq 1, \tag{1.1}$$

with initial conditions $h_0 = a$ and $h_1 = b$. This sequence like was introduced by Horadam [1, 2], and it generalizes many sequences with the characteristic equation of recurrence relation of form $x^2 - px - q = 0$. More general results about the Horadam sequence can be found in [1, 3].

In Equation (1.1), if we let $p = 11; q = -10; a = 0$; and $b = 1$; then the Horadam sequence is specified in the repunit sequence. The repunit numbers $\{r_n\}_{n\geq 0}$ are the terms of the sequence $\{0, 1, 11, 111, 1111, 1111, \ldots\}$ where each term satisfies the recursive formula $r_{n+1} = 10r_n + 1$ for all $n \geq 1$ and $r_0 = 0$, the sequence $A002275$ in OEIS [4]. Santos and Costa [5] showed that this sequence too satisfies the Horadam recursive recurrence

$$r_{n+1} = 11r_n - 10r_{n-1}, \text{for all } n \geq 1, \tag{1.2}$$

with initial conditions $r_0 = 0$ and $r_1 = 1$. Some work too explore the connections of repunit sequence $\{r_n\}_{n\geq 0}$ with the Fibonacci-type or Lucas-type sequence, again a Horadam-type sequence, $l_{n+1} = pl_n - ql_{n-1}$, for all $n \geq 1$, p and q are fixed, and l_1 and l_0 are given, see instance [6, 7].

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This article is organized into distinct sections: Section 2 revisits some concepts related to sequence repunits, such as Horadam recurrence and Binet's formula. Section 3 explores classical identities for repunit sequence with integer indices, highlighting the Tagiuri-Vajda, D'Ocagne, Catalan, Cassini, and Gelin-Cesàro identities. Section 4 shows the relationships between repunits and powers with natural exponents, including proof that no repunit number can be a power of any natural number. In Section 5 investigates the divisibility relation between a prime p and the composite repunit r_{p^n-1}. Finally, Section 6 we present the partial sums of the terms in the sequence. Even though the repunit sequence is a Horadam-type sequence, this is the first work to determine the Tagiuri-Vajda, and Gelin-Cesàro identities of the repunit sequence.

2 Repunit numbers and Binet formula

In this section, we establish some recurrence relations and the Binet formula for repunit sequence for all integers n.

See that the difference equation associated with the sequence of repunit $\{r_n\}_{n\geq 0}$ is

$$r_{n+1} = 11r_n - 10r_{n-1}, \text{ for all } n \geq 1, \tag{2.1}$$

which has as its Horadam-type characteristic equation $r^2 - 11r + 10 = 0$, and its real roots are $r_1 = 10$ and $r_2 = 1$. According [8] the equation $r^2 + pr + q = 0$ has distinct roots r_1 and r_2, and then the sequences $r_n = c_1(r_1)^n + c_2(r_2)^n$, for $n \geq 0$, and with c_1, c_2 real numbers are solutions of Equation (2.1). Let us determine the constants c_1 and c_2, considering that $r_0 = 0$ and $r_1 = 1$, and we obtain the linear system,

$$\begin{cases} 0 = c_1 + c_2 \\ 1 = 10c_1 + c_2 \ . \end{cases}$$

We find $c_1 = \frac{1}{9}$ and $c_2 = -\frac{1}{9}$. So we have that

$$r_n = \frac{10^n - 1}{9} \ , \text{ for all } n \geq 0. \tag{2.2}$$

The Equation (2.2) presents the classic and well-known Binet's formula for the sequence of repunit $\{r_n\}_{n\geq 0}$, see the references [9–11].

In Costa and others [12] the repunit sequence $\{r_n\}_{n\geq 0}$ the was extended to negative subscripts this way. Let $n \geq 1$, then the negative index n-th repunit numbers is defined as

$$r_{-n} = -\frac{r_n}{10^n} \ . \tag{2.3}$$

It follows from the definition that repunit sequence with negative index is the set of elements given by

$$\{r_{-n}\}_{n\geq 1} = \left\{-\frac{1}{10},\ -\frac{11}{10^2},\ -\frac{111}{10^3}, \ldots\right\} = \{-0,1;\ -0,11;\ -0,111, \ldots\}.$$

The first few repunit numbers with negative subscript are given in the following Table 1, with $-8 \leq n \leq -1$:

n	-8	-7	-6	-5	-4	-3	-2	-1
r_n	- 0,11111111	-0,1111111	-0,111111	-0,11111	-0,1111	-0,111	-0,11	-0,1

Table 1. Repunit numbers at negative index

According [12], observation of Table 1, we conclude that the repunit sequence with negative index satisfies the recurrence relation

$$r_{-(n+1)} = \frac{11}{10} r_{-n} - \frac{1}{10} r_{-(n-1)} \text{ with } r_{-1} = -0,1 \text{ and } r_{-2} = -0,11 \; ; \qquad (2.4)$$

for $n = 1, 2, 3, \ \ldots$.

Note that the recurrence $r_{-(n+1)} = \frac{11}{10} r_{-n} - \frac{1}{10} r_{-(n-1)}$ has Horadam-type characteristic equation given by

$$r^2 - \frac{11}{10} r + \frac{1}{10} = 0 \; , \qquad (2.5)$$

whose roots are $r_1 = \frac{1}{10}$ and $r_2 = 1$. We find $c_1 = \frac{-1}{9}$ and $c_2 = \frac{1}{9}$. Then, the Binet formula is as follows.

Proposition 2.1. [12] (Binet's formula) For all $n \in \mathbb{N}$, we have

$$r_{-n} = -\frac{10^n - 1}{9 \cdot 10^n} \; . \qquad (2.6)$$

3 Some Classical Identities

In this section, we establish some classical identities for the repunit sequence for all integers n, for example, the Tagiuri-Vajda, Catalan, Cassini, and d'Ocganes identities. The Tagiuri-Vajda and Gelin-Cesàro identities are presented for the first time in this work. The identities of Catan, Cassini, and d'Ocganes have already appeared in some previous work, however the demonstration presented is different from the reference indicated.

3.1 Identities with subscript positive

First the Tagiuri-Vajda's Identity:

Theorem 3.1. *Let m, s, k be any natural numbers. We have*

$$r_{m+s} r_{m+k} - r_m r_{m+s+k} \;=\; 10^m r_s r_k .$$

Proof. Using Equation (2.2) again we obtain that

$$\begin{aligned}
& r_{m+s} r_{m+k} - r_m r_{m+s+k} \\
= \;& \left(\frac{10^{m+s} - 1}{9}\right)\left(\frac{10^{m+k} - 1}{9}\right) - \left(\frac{10^m - 1}{9}\right)\left(\frac{10^{m+s+k} - 1}{9}\right) \\
= \;& \frac{10^{m+s+k} - 10^{m+k} - 10^{m+s} + 10^m}{81} = \frac{(10^{m+k} - 10^m)(10^s - 1)}{81} \\
= \;& 10^m \left(\frac{10^k - 1}{9}\right)\left(\frac{10^s - 1}{9}\right),
\end{aligned}$$

and we have the validity of the result. ∎

The other identities will follow as a consequence of the Tagiuri-Vajda's Identity, Theorem 3.1, as we will see below.

Proposition 3.2. [5] (d'Ocagne's Identity) Let m, n be any natural. For $m \geq n$ we have

$$r_m r_{n+1} - r_{m+1} r_n \;=\; 10^n r_{m-n} \;.$$

Proof. First, we consider $k = n - m$ and $s = 1$ in Theorem 3.1, so

$$r_{m+1} r_n - r_m r_{n+1} \;=\; 10^m r_{n-m} \;.$$

Note that $n - m < 0$, using to Equation 2.3 we obtain that

$$\begin{aligned} r_{m+1} r_n - r_m r_{n+1} &= 10^m \left(-\frac{r_{m-n}}{10^{m-n}} \right) \\ &= -10^n r_{m-n} \;, \end{aligned}$$

and we get the result. ∎

Similar to Proposition 3.2 we have the Catalan's Identity.

Proposition 3.3. [5]. Let m, n be any natural. For $m \geq n$ we have

$$(r_m)^2 - r_{m-n} r_{m+n} \;=\; 10^{m-n} \cdot (r_n)^2 \;.$$

Proof. Just take $s = n$ and $k = -n$ in Theorem 3.1 we have that

$$\begin{aligned} r_{m+n} r_{m-n} - (r_m)^2 &= 10^m \left(\frac{10^{-n} - 1}{9} \right) \left(\frac{10^n - 1}{9} \right) \\ &= 10^m \left(\frac{\frac{1-10^n}{10^n}}{9} \right) \left(\frac{10^n - 1}{9} \right) \\ &= 10^m \left(-\frac{10^n - 1}{10^n \cdot 9} \right) \left(\frac{10^n - 1}{9} \right) \\ &= -10^{m-n} \left(\frac{10^n - 1}{9} \right)^2 \;. \end{aligned}$$

Now, the result will follow with the help of Equations (2.2).

∎

Making $n = 1$, without much effort, follows directly from Proposition 3.3 that:

Proposition 3.4. [5]. (Cassini's Identity) For all $m \geq 1$, we have

$$(r_m)^2 - r_{m-1} r_{m+1} \;=\; 10^{m-1} \;.$$

The following result presents an interesting application of Cassini's Identity, as shown below:

Proposition 3.5. For any positive integer m, the Diophantine equation $x^2 - 11xy + 10y^2 = 10^{m-1}$ has infinitely many solutions.

Proof. By combining Cassini's Identity (Proposition 3.4) with Equation (1.2), we obtain the following equation: $(r_m)^2 - 11r_{m-1}r_m + 10(r_{m-1})^2 = 10^{m+1}$. Setting $r_m = x$ and $r_{m-1} = y$ yields the desired result. □

Now, we present the Gelin-Cesàro identity for the repunit sequence below.

Proposition 3.6. Let m be any natural number. Wave

$$r_{m-2}r_{m-1}r_{m+1}r_{m+2} = r_m^4 - 111 \cdot 10^{m-2} \cdot r_m^2 + 121 \cdot 10^{2m-3}$$

Proof. By setting $n = 2$ in Proposition 3.3, we obtain the following equation:

$$(r_m)^2 - r_{m+2}r_{m-2} = 121 \cdot 10^{m-2} \ . \tag{3.1}$$

Multiplying Equation (3.1) with Cassini's Identity, Proposition 3.4, leads to the desired result □

To conclude this subsection, we present an interesting result that explores combinations of some terms in the repunit sequence. This result has similarities with the Tagiuri-Vajda, Catalan, Cassini, and d'Ocagne identities discussed previously. Before an auxiliary result.

Lemma 3.7. *Let m be any natural. We have*

$$r_{m+2}r_{m+3}r_{m+4} = r_{m+1}r_{m+2}r_{m+6} + 1221 \cdot 10^{m+1}r_{m+2}.$$

Proof. Using the Equation 2.2, we get

$$\begin{aligned} r_{m+2}r_{m+3}r_{m+4} &= \left(\frac{10^{m+3}-1}{9}\right)\left(\frac{10^{m+4}-1}{9}\right)r_{m+2} \\ &= \frac{10^{2m+7} - 10^{m+3} - 10^{m+4} + 1}{81} r_{m+2} \\ &= \frac{10^{m+1}(10^{m+6} - 10^2 - 10^3) + 1}{81} r_{m+2} \ , \end{aligned}$$

the same way,

$$\begin{aligned} r_{m+1}r_{m+6}r_{m+2} &= \left(\frac{10^{m+1}-1}{9}\right)\left(\frac{10^{m+6}-1}{9}\right)r_{m+2} \\ &= \frac{10^{2m+7} - 10^{m+6} - 10^{m+1} + 1}{81} r_{m+2} \\ &= \frac{10^{m+1}(10^{m+6} - 10^5 - 1) + 1}{81} r_{m+2} \ . \end{aligned}$$

So

$$
\begin{aligned}
r_{m+2}(r_{m+3}r_{m+4} - r_{m+1}r_{m+6}) =& \frac{10^{m+1}(10^5 - 10^2 - 10^2 + 1)}{81} r_{m+2} \\
=& 10^{m+1}\left(\frac{10^2-1}{9}\right)\left(\frac{10^3-1}{9}\right) r_{m+2} \\
=& 1221 \cdot 10^{m+1} r_{m+2}
\end{aligned}
$$

□

Proposition 3.8. Let m be any natural. We have

$$(r_{m+3})^3 - r_{m+1}r_{m+2}r_{m+6} = 10^{m+1}(1331r_{n+2} - 100r_{n+1}) \ .$$

Proof. By Cassini's Identity, Proposition 3.4, we have that $(r_{m+3})^2 - r_{m+2}r_{m+4} = 10^{m+2}$. So $(r_{m+3})^3 - r_{m+2}r_{m+3}r_{m+4} = 10^{m+2}r_{m+3}$. According Lemma 3.7, we get

$$
\begin{aligned}
(r_{m+3})^3 - r_{m+2}r_{m+3}r_{m+4} &= 10^{m+2}r_{m+3} \\
(r_{m+3})^3 - r_{m+1}r_{m+2}r_{m+6} &= 1221 \cdot 10^{m+1}r_{m+2} + 10^{m+2}r_{m+3} \\
&= 10^{m+1}(1221r_{m+2} + 10r_{m+3}) \\
&= 10^{m+1}(1331r_{n+2} - 100r_{n+1}) \ .
\end{aligned}
$$

Since $r_{m+3} = 11r_{m+2} - 10r_{m+1}$, Equation (2.1). □

3.2 Identities with subscript negative

Similarly, in the following result, we show the Tagiuri-Vajda identity for the negative subscripts of r_n.

Theorem 3.9. *Let m, s, k be any natural numbers. We have*

$$r_{-(m+s)}r_{-(m+k)} - r_{-m}r_{-(m+s+k)} \;=\; 10^{-m}r_{-s}r_{-k} \ .$$

Proof. Using Binet's Formula defined by the Equation (2.6), we obtain the following expression:

$$
\begin{aligned}
& r_{-(m+s)}r_{-(m+k)} - r_{-m}r_{-(m+s+k)} \\
=& \left(\frac{10^{m+s}-1}{9\cdot 10^{m+s}}\right)\left(\frac{10^{m+k}-1}{9\cdot 10^{m+k}}\right) - \left(\frac{10^{m}-1}{9\cdot 10^{m}}\right)\left(\frac{10^{m+s+k}-1}{9\cdot 10^{m+s+k}}\right) \\
=& \frac{10^{m+s+k} - 10^{m+k} - 10^{m+s} + 10^m}{81\cdot 10^{2m+s+k}} = \frac{(10^{m+k}-10^m)(10^s-1)}{81\cdot 10^{2m+s+k}} \\
=& 10^{-m}\left(-\frac{10^k-1}{9\cdot 10^k}\right) r_{-s} = 10^{-m}r_{-k}r_{-s} \ ,
\end{aligned}
$$

and we have the validity of the result. □

Again the other identities will follow as a consequence of the Tagiuri-Vajda's Identity, Theorem 3.9, as we will see below. Firt, we present the D'Ocagne Identity for the negative indices of r_n.

Proposition 3.10. [12] Let $m,\ n$ be any natural. For $m \geq n$ we have

$$r_{-(m+1)}r_{-n} - r_{-m}r_{-(n+1)} = \frac{r_{m-n}}{10^{m+1}} \ .$$

Proof. he result follows directly from Theorem 3.1, by substituting $k = n - m$ and $s = 1$. □

Similar to Proposition 3.10 we have the Catalan's Identity.

Proposition 3.11. [12] Let $m,\ n$ be any natural. For $m \geq n$ we have

$$(r_{-m})^2 - r_{-(m-n)}r_{-(m+n)} = \frac{(r_n)^2}{10^{(m-n)}} \ .$$

Proof. Just take $s = n$ and $k = -n$ in Theorem 3.9 and the result will follow. □

When substituting $n = 1$ in Proposition 3.11, we obtain

Proposition 3.12. [12] (Cassini's Identity) For all $m \geq 1$, we have

$$(r_{-m})^2 - r_{-(m-1)}r_{-(m+1)} = 10^{-(m-1)} \ .$$

We conclude this section we present the Gelin-Cesàro identity can also be extended to negative indices. The resulting identity for negative indices is presented below:

Proposition 3.13. For all $m \geq 1$, we have

$$r_{-(m-2)}r_{-(m-1)}r_{-(m+1)}r_{-(m+2)} = r_{-m}^4 - 111 \cdot 10^{-(m-2)} \cdot r_{-m}^2 + 121 \cdot 10^{-(2m+3)} \ .$$

Proof. The desired result can be obtained by simply multiplying the results of Proposition 3.11 and Proposition 3.12, setting $n = 2$ in both cases. □

Lemma 3.14. *Let m be any natural number. We have*

$$r_{-(m+2)}r_{-(m+3)}r_{-(m+4)} = r_{-(m+1)}r_{-(m+2)}r_{-(m+6)} + 1221 \cdot 10^{-(m+6)}r_{-(m+2)}.$$

Proof. Using the Equation 2.6, we get

$$\begin{aligned} r_{-(m+2)}r_{-(m+3)}r_{-(m+4)} &= \left(\frac{10^{m+3}-1}{9 \cdot 10^{m+3}}\right)\left(\frac{10^{m+4}-1}{9 \cdot 10^{m+4}}\right) r_{-(m+2)} \\ &= \frac{10^{2m+7} - 10^{m+3} - 10^{m+4} + 1}{81 \cdot 10^{2m+7}} r_{-(m+2)} \\ &= \frac{10^{m+1}(10^{m+6} - 10^2 - 10^3) + 1}{81 \cdot 10^{2m+7}} r_{-(m+2)} \ , \end{aligned}$$

the same way,

$$\begin{aligned} r_{-(m+1)}r_{-(m+2)}r_{-(m+6)} =& \left(\frac{10^{m+1}-1}{9\cdot 10^{m+1}}\right)\left(\frac{10^{m+6}-1}{9\cdot 10^{m+6}}\right)r_{-(m+2)} \\ =& \frac{10^{2m+7}-10^{m+6}-10^{m+1}+1}{81\cdot 10^{2m+7}}r_{-(m+2)} \\ =& \frac{10^{m+1}(10^{m+6}-10^{5}-1)+1}{81\cdot 10^{2m+7}}r_{-(m+2)} \\ . \end{aligned}$$

So

$$\begin{aligned} r_{-(m+2)}(r_{-(m+3)}r_{-(m+4)}-r_{-(m+1)}r_{-(m+6)}) =& \frac{10^{m+1}(10^{5}-10^{2}-10^{2}+1)}{81\cdot 10^{2m+7}}r_{-(m+2)} \\ =& \frac{10^{m+1}}{10^{2m+7}}\left(\frac{10^{2}-1}{9}\right)\left(\frac{10^{3}-1}{9}\right)r_{-(m+2)} \\ =& 1221\cdot 10^{-(m+6)}r_{-(m+2)} \end{aligned}$$

□

In conclusion, we demonstrate that the following identity holds for negative indices:

Proposition 3.15. Let n be any natural number. Wave

$$r^{3}_{-(n+3)}-r_{-(n+1)}r_{-(n+2)}r_{-(n+6)} = 10^{-(m+1)}\left(\frac{12221}{100000}r_{-(n+2)}-\frac{1}{100}r_{-(n+1)}\right) .$$

Proof. By Cassini's Identity, Proposition 3.12, we have that $(r_{-(m+3)})^{2}-r_{-(m+2)}r_{-(m+4)} = 10^{(-m+2)}$. So $(r_{-(m+3)})^{3}-r_{-(m+2)}r_{-(m+3)}r_{-(m+4)} = 10^{-(m+2)}r_{-(m+3)}$.

According Lemma 3.14, we get

$$\begin{aligned} (r_{-(m+3)})^{3}-r_{-(m+2)}r_{-(m+3)}r_{-(m+4)} &= 10^{-(m+2)}r_{-(m+3)} \\ (r_{-(m+3)})^{3}-r_{-(m+1)}r_{-(m+2)}r_{-(m+6)} &= \frac{1221}{100000}\cdot 10^{-(m+1)}r_{-(m+2)}+10^{-(m+2)}r_{-(m+3)} \\ &= 10^{-(m+1)}\left(\frac{1221}{100000}r_{m+2}+\frac{1}{10}r_{m+3}\right) \\ &= 10^{-(m+1)}\left(\frac{12221}{100000}r_{-(n+2)}-\frac{1}{100}r_{-(n+1)}\right) . \end{aligned}$$

Since $r_{-(m+3)} = \frac{11}{10}r_{-(m+2)}-\frac{1}{10}r_{-(m+1)}$, Equation (2.4)

□

4 Repunit number are not powers

We remember that a natural number m written as a power of natural numbers, if there are natural numbers a and k such that $m = a^{k}$.

In [5] it is shown that for $n \geq 2$, no repunit r_n is an even power or a sum of two even powers. In this section, we will extend this result to any natural number $n \geq 2$.

The main result of this section is

Theorem 4.1. *For $n \geq 2$, no r_n is a power of natural numbers.*

The Theorem 4.1 is a direct consequence of the next two results.

Proposition 4.2. [5] For $n \geq 2$, neither r_n is a even power or a sum of two even powers.

Proposition 4.3. For $n \geq 3$, neither r_n is an odd power.

The proof of Propositions 4.2 can be consulted at [5], and the Proposition 4.3 will be demonstrated in subsection below.

4.1 Odd power

For every natural number m, let $\varphi(m)$ be the quantity of natural numbers less than or equal to m that are relatively prime to m, meaning $\gcd(a, m) = 1$ for all $a \leq m$, where $\gcd(a, b)$ is the greatest common divisor between the numbers a e b. The function $\varphi(m)$ is known as the Euler's totient.

To present the result, we will make use of the following two auxiliary results.

Lemma 4.4. *[13] The equation*

$$\frac{x^n - 1}{x - 1} = y^m, \;\; x, y, m, n \in \mathbb{N}, \;\; x > 1, \;\; y > 1, \;\; m > 2, \;\; n > 1 \;,$$

has no solution $(x,\; y,\; m,\; n)$ *satisfying* $\gcd\big(x\varphi(x), m\big) = 1$, *where* $\varphi(m)$ *is Euler's function of* m.

Lemma 4.5. *[5, 14] For $n \geq 2$, no repunit r_n is a fifth power.*

Now let us extend the Lemma above to all odd prime $p \neq 5$.

Proposition 4.6. Let $n \geq 2$ and $p \neq 5$ be an odd prime. No repunit r_n is a power p.

Proof. By the hypothesis, we have $p > 2$ a prime number. For all $n \geq 1$, let r_n be a repunit number. Using the Binet's formula, Equation 2.2, we get that

$$r_n = \frac{10^n - 1}{9} = 1 + 10 + 10^2 + 10^3 + \ldots + 10^{n-1} \;.$$

According to Lemma 4.4, we can conclude that the Diophantine equation

$$1 + 10 + 10^2 + 10^3 + \ldots + 10^{n-1} = y^p \;,$$

does not have integer solutions for any $n,\ y$ given that $\gcd\big(10\varphi(10), p\big) = \gcd\big(40, p\big) = 1$, since $\varphi(10) = 4$, and $p \neq 5$. $\square$

Finally, we must show that r_n cannot be expressed in the form a^m for any natural numbers a and m, where m is odd. So let's go to:
proof of Proposition 4.3
Let n be a natural number. Consider the repunit number r_n, and a^m any positive integers a and m where m is odd. Now, will analyze the case where $m = pz$ for an odd number z. Suppose that, for some $n \geq 2$, we have $r_n = a^{pz}$. This would imply that r_n is a p power, since $r_n = (a^z)^p$ which contradicts Lemma 4.5 if $p = 5$, or contradicts Proposition 4.6 if $p \neq 5$, and we conclude the demonstration.

5 Factor p of the repunit number

In [11, 15], we have that for any prime $p > 5$, then p is a divisor of some repunit, precisely p divides r_{p-1}. In this section, let us extend the result showing that p is a divisor of r_{p^n-1}. It is also noteworthy that in [10] presenting also a characterization for a prime factor of a non-prime repunit type r_p.

In order to verify the result later, we need an auxiliary result, which we present below:

Lemma 5.1. *[16] If a and b are integers and n is a natural number, then $a - b$ divides $a^n - b^n$.*

Lemma 5.2. *[16](Euler-Fermat's Theorem) Let a and m be natural numbers. If $\gcd(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 \pmod{m}$.*

Theorem 5.3. *Let n be a positive integer and p a prime with $p > 5$. Then p divides r_{p^n-1}.*

Proof. It follows from Lemma 5.1 that $p-1$ divides $p^n - 1$, that is, $p^n - 1 = (p-1)k$ for some integer k. Since $p > 5$, then $(10^k, p) = 1$. Therefore, by Euler-Fermat's Theorem, Lemma 5.2, we have that $(10^k)^{\varphi(p)} \equiv 1 \pmod{p}$. Since $\varphi(p) = p - 1$, we have that $(10^k)^{(p-1)} \equiv 10^{p^n-1} \equiv 1 \pmod{p}$ and the result follows. □

Making $n = 1$ in Proposition 5.3, have we:

Corollary 5.4. [15] Let p be a prime number, $p > 5$, then p divides r_{p-1}.

6 Sum formulas

In this section, we present results on partial sums of terms of the repunit sequence with n integers. Initially, consider the sequence of partial sums $\sum_{k=0}^{n} r_k = r_0+r_1+r_2+\cdots+r_n$, for $n \geq 0$, where $\{r_n\}_{n\geq0}$ is the repunit sequence. We have two auxiliary result.

Proposition 6.1. [12] Let $\{r_n\}_{n\geq1}$ be the repunit sequence, then

$$\begin{aligned}
\text{(a)} \sum_{k=0}^{n} r_k &= \frac{10r_n - n}{9}, \\
\text{(b)} \sum_{k=0}^{n} r_{2k} &= \frac{10^2 r_{2n} - nr_2}{99}, \\
\text{(c)} \sum_{k=0}^{n} r_{2k+1} &= \frac{r_{2n+3} - (n+1)r_2}{99}.
\end{aligned}$$

Proposition 6.2. [12] Let $\{r_{-n}\}_{n\geq0}$ be the repunit negative sequence, then

$$\begin{aligned}
\text{(a)} \sum_{k=0}^{n} r_{-k} &= -\frac{n - r_{-n}}{9}, \\
\text{(b)} \sum_{k=0}^{n} r_{-2k} &= -\frac{nr_2 - r_{-2n}}{99}, \\
\text{(c)} \sum_{2k=0}^{n} r_{-(2k+1)} &= -\frac{(n+1)r_2 - r_{-(2n+1)}}{99}.
\end{aligned}$$

Now, consider the sequence of alternating partial sums $\sum_{k=0}^{n}(-1)^k r_k = r_0 - r_1 + r_2 - r_3 + \cdots + (-1)^n r_n$, for $n \geq 0$, being $\{r_n\}_{n\geq 0}$ the repunit sequence.

Proposition 6.3. Let $\{r_n\}_{n\geq 0}$ be the repunit sequence, then

$$\begin{array}{llll} \text{(a)} & \displaystyle\sum_{k=0}^{n}(-1)^k r_k & = \frac{10^2 r_{2n} - r_{2n+3} + r_2}{99}, & \text{if last term is negative ;} \\ \text{(b)} & \displaystyle\sum_{k=0}^{n}(-1)^k r_k & = \frac{10^2 r_{2n} - r_{2n+1}}{99}, & \text{if last term is positive .} \end{array}$$

Proof. (a) First consider that last term is negative, so

$$\begin{aligned} \sum_{k=0}^{2n+1}(-1)^k r_k &= r_0 - r_1 + r_2 - r_3 + \cdots + r_{2n} - r_{2n+1} \\ &= (r_0 + r_2 + \cdots + r_{2n}) - (r_1 + r_3 + \cdots + r_{2n+1}) \\ &= \sum_{k=0}^{n} r_{2k} - \sum_{k=0}^{n} r_{2k+1} \end{aligned}$$

According to the Proposition 6.1, items (b) and (c), it follows that:

$$\begin{aligned} \sum_{k=0}^{2n+1}(-1)^k r_k &= \sum_{k=0}^{n} r_{2k} - \sum_{k=0}^{n} r_{2k+1} \\ &= \frac{10^2 r_{2n} - n r_2}{99} - \frac{r_{2n+3} - (n+1) r_2}{99} \\ &= \frac{10^2 r_{2n} - r_{2n+3} + r_2}{99} . \end{aligned}$$

(b) In which case that last term is positive, so

$$\begin{aligned} \sum_{k=0}^{2(n+1)}(-1)^k r_k &= r_0 - r_1 + r_2 - r_3 + \cdots + r_{2n} - r_{2n+1} + r_{2(n+1)} \\ &= (r_0 + r_2 + \cdots + r_{2n} + r_{2(n+1)}) - (r_1 + r_3 + \cdots + r_{2n+1}) \\ &= \sum_{k=0}^{n+1} r_{2k} - \sum_{k=0}^{n} r_{2k+1} . \end{aligned}$$

As in item (a), apply the Proposition 6.1. □

Proposition 6.4. Let $\{r_{-n}\}_{n\geq 0}$ be the repunit negative sequence, then

$$\begin{array}{llll} \text{(a)} & \displaystyle\sum_{k=0}^{n}(-1)^k r_{-k} & = \frac{r_2 + r_{-2n} - r_{-2(n+1)}}{99}, & \text{if last term is negative ;} \\ \text{(b)} & \displaystyle\sum_{k=0}^{n}(-1)^k r_{-k} & = \frac{r_{-2n} - r_{-2(n+1)}}{99}, & \text{if last term is positive .} \end{array}$$

Proof. (a) First consider that last term is negative, so

$$\begin{aligned}
\sum_{k=0}^{2n+1}(-1)^k r_{-k} &= r_0 - r_{-1} + r_{-2} - r_{-3} + \cdots + r_{-2n} - r_{-(2n+1)} \\
&= (r_0 + r_{-2} + \cdots + r_{-2n}) - (r_{-1} + r_{-3} + \cdots + r_{-(2n+1)}) \\
&= \sum_{k=0}^{n} r_{-2k} - \sum_{k=0}^{n} r_{-(2k+1)}
\end{aligned}$$

According to the Proposition 6.2, items (b) and (c), it follows that:

$$\begin{aligned}
\sum_{k=0}^{2n+1}(-1)^k r_{-k} &= -\frac{nr_2 - r_{-2n}}{99} + \frac{(n+1)r_2 - r_{-(2n+1)}}{99} \\
&= \frac{r_2 + r_{-2n} - r_{-2(n+1)}}{99}\,.
\end{aligned}$$

(b) In which case that last term is positive, so

$$\begin{aligned}
\sum_{k=0}^{2(n+1)}(-1)^k r_{-k} &= r_0 - r_{-1} + r_{-2} - r_{-3} + \cdots + r_{-2n} - r_{-(2n+1)} + r_{-2(n+1)} \\
&= \sum_{k=0}^{n+1} r_{-2k} - \sum_{k=0}^{n} r_{-(2k+1)}
\end{aligned}$$

As in item (a), apply the Proposition 6.2. □

7 Conclusion

This paper discusses properties regarding the repunit sequence, a Horadam-type sequence $\{h_n\}_{n\geq 0}$. We primarily delve into the classical identities of the repunit sequence with integer indices. Additionally, we establish that no repunit can be expressed as an odd perfect power, thereby extending the previously known result that repunits cannot be written as powers with even exponents, thus concluding that repunits cannot be perfect powers. By shedding light on these findings, we aim to inspire other explorations of this number class. Notably, some aspects of these investigations appear to be pioneering, suggesting that while the results are obtained through elementary mathematical methods, they may offer original insights, potentially enriching the field's understanding.

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