*INTERMATHS, VOL. 5, NO. 2 (**2024**)*, **80–92**
https://doi.org/10.22481/intermaths.v5i2.15554

On *One-Zero* numbers: A new Horadam-type sequence

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Abstract: In this paper, we present a new sequence of Horadam-type, which we call the *One-Zero* sequence. We study the recurrence equation and show the Binet formula. The aim of this study is to examine the properties of the aforementioned sequence. To this end, we have analyzed several classical identities, including the Tagiuri-Vajda and the Gelin-Cesàro identities. Additionally, we determine the partial sum of the terms of the *One-Zero* sequence.

Keywords: Horadam-type sequence; *One-Zero* sequence; Tagiuri-Vajda Identity, Partial sum.

Classification MSC: primary 11B37; secondary 11B39, 11B83.

1 Introduction

For a non-negative integer n, consider the Horadam sequence $\{h_n\}_{n\geq 0}$ defined by the second-order recurrence relation, where p and q are fixed integers:

$$h_{n+1} = ph_n + qh_{n-1} \quad \text{for all } n \geq 1, \tag{1.1}$$

with initial terms $h_0 = a$ and $h_1 = b$. This sequence was introduced by Horadam [1, 2], generalizes many sequences defined by a recurrence relation of the form $x^2 - px - q = 0$. For more comprehensive results on the Horadam sequence, see [2, 3].

The sequence *One-Zero* consists of natural numbers that are represented in the decimal system only by the digits 1 (unit-one) and 0 (zero) alternately starting and ending with 1, represented by the set $\{U_n\}_{n\geq 1} = \{1,\ 101,\ 10101, \dots, \}$, the id-number $A094028$ in OEIS [4]. Let us begin with the elementary observation that the sequence $\{U_n\}_{n\geq 1}$ satisfies the non-homogeneous linear recurrence:

$$U_{n+1} = 100U_n + 1, \text{ with } U_1 = 1 \text{ and } n \geq 1 \ . \tag{1.2}$$

An alternative way to express the equation (1.2) is

$$U_n = 100U_{n-1} + 1 \ . \tag{1.3}$$

By subtracting the equations (1.2) and (1.3) we get a homogeneous recurrence relation,

$$U_{n+1} = 101U_n - 100U_{n-1} \ , \text{ with } U_0 = 0 \text{ and } U_1 = 1, \text{ for all } n \geq 1 \ , \tag{1.4}$$

where U_n represents the n-th *One-Zero* numbers, for convenience, we will use $U_0 = 0$.

In Equation (1.1), if we assume the values of $p = 101, q = -100, a = 0$, and $b = 1$, then the Horadam sequence is specified in the *One-Zero* sequence. It can be observed that the *One-Zero*

Submitted 10 November 2024; Accepted 23 December 2024; Available online: 31 December 2024.

numbers, represented by the sequence $\{U_n\}_{n\geq 0}$, satisfy the Horadam recursive recurrence, as defined by Equation (1.4).

These numbers are referred to as *smoothly undulating*, and various arithmetic properties involving divisibility, primality, and the existence of perfect squares are discussed in [5–10], among other sources.

In this section we present the definition of the *One-Zero* sequence and identify it as a Horadam-type sequence. The structure of this paper is organized into four additional sections, outlined as follows. In Section 2, we establish recurrence relations and a Binet's formula for the sequence *One-Zero*, showing an expression for understanding and efficiently calculating its terms for any integer n. As well as determining the generating functions for this sequence. In Section 3, using Binet's formula, we establish some identities for the *One-Zero* sequence for all integers n. This is the first work to determine and display identities for this sequence. Finally, summation formulas involving the *One-Zero* numbers are presented in Section 4. We conclude with some final considerations and state some future work on this topic.

2 *One-Zero* numbers and the Binet formula

In this section, we derive the recurrence relation and formulate the Binet expression for the *One-Zero* sequence, applicable to all integers n. This provides a comprehensive framework for understanding and systematically calculating terms in the sequence.

See that the difference equation associated with the *One-Zero* sequence $\{U_n\}_{n\geq 0}$ is

$$U_n = 101U_{n-1} - 100U_{n-2},\ \text{for all } n \geq 2,\ \text{with } U_0 = 0 \text{ and } U_1 = 1\ ; \tag{2.1}$$

which has as its Horadam-type characteristic equation $x^2 - 101x + 100 = 0$, and its real roots are $x_1 = 100$ and $x_2 = 1$. Following [11, 12], if the equation $x^2 - px + q = 0$ has distinct roots x_1 and x_2, then, for $n \geq 0$, the expression

$$x_n = c_1(x_1)^n + c_2(x_2)^n\ , \tag{2.2}$$

with c_1, c_2 real numbers are solutions of Equation (2.1). Let us determine the constants c_1 and c_2. With the initial terms $U_0 = 0$ and $U_1 = 1$, we obtain the following linear system:

$$\begin{cases} 0 = c_1 + c_2 \\ 1 = 100c_1 + c_2\ . \end{cases}$$

We find that $c_1 = \frac{1}{99}$ and $c_2 = -\frac{1}{99}$. Substituting these values into Equation (2.2) yields:

$$U_n = \frac{10^{2n} - 1}{99}\ ,\ \text{for all } n \geq 0.$$

This verifies the following result:

Proposition 2.1. *For all $n \geq 0$ integer numbers, we have*

$$U_n = \frac{10^{2n} - 1}{99}\ , \tag{2.3}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

The proposition above, Equation (2.3), represents the classic Binet formula for the *One-Zero* sequence $\{U_n\}_{n\geq 0}$.

In order to extend the *One-Zero* sequence to incorporate negative subscripts, it is necessary to set $n = 1$ in the Equation (2.1), as follows:

$$\begin{aligned} U_1 &= 101U_0 - 100U_{-1} \\ 1 &= 0 - 100U_{-1}; \end{aligned}$$

or equivalent,

$$U_{-1} = -\frac{1}{100} = -\frac{U_1}{100}\ ;$$

in the same way, let's make $n = 0$,

$$U_0 = 101U_{-1} - 100U_{-2}$$
$$0 = -\frac{101}{100} - 100O{-2} \; ;$$

or equivalent,

$$U_{-2} = -\frac{101}{100^2} = -\frac{U_2}{100^2} \; .$$

Having stated it, let's define the negative index terms of this sequence.

Definition 2.2. *For any integer $n \geq 0$, the One-Zero sequence for negative indexes is defined as follows:*

$$U_{-n} = -\frac{U_n}{10^{2n}} \; . \tag{2.4}$$

This definition makes sense because by making $n = -k$ in Equation (2.3), we get

$$U_{-k} = \frac{10^{-2k} - 1}{99} = -\frac{10^{2k} - 1}{99 \cdot 10^{2k}} \; .$$

The Binet formula above for the *One-Zero* sequence $\{U_{-n}\}_{n\geq 0}$ with negative subscripts, is a direct consequence from Equations (2.3) and (2.4).

Proposition 2.3. *For all integer numbers $n \geq 0$, we have*

$$U_{-n} = -\frac{10^{2n} - 1}{99 \cdot 10^{2n}} \; , \tag{2.5}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

According to definition, a *One-Zero* sequence with negative index is constituted by the set of elements given by

$$\{U_{-n}\}_{n\geq 1} = \left\{ -\frac{1}{10^2}, \; -\frac{101}{10^4}, \; -\frac{10101}{10^6}, -\frac{1010101}{10^8}, \ldots, \right\} \; .$$

We can rewrite the recurrence relation Equation (2.1) for all n integer, the *One-Zero* sequence $\{U_n\}_{n\in\mathbb{Z}}$ satisfies:

$$U_n = 101U_{n-1} - 100U_{n-2}, \text{ for all } n \geq 2, \text{ with } U_0 = 0 \text{ and } U_1 = 1 \; .$$

Again, the *One-Zero* sequence $\{U_{-n}\}_{n\geq 0}$ with negative subscripts satisfies:

Proposition 2.4. *For all $n > 0$, the following recurrence holds*

$$U_{-(n+1)} = \frac{101U_{-n} - U_{-(n-1)}}{10^2} \; , \tag{2.6}$$

where $\{U_n\}_{n\in\mathbb{Z}}$ is the One-Zero sequence.

Proof. Note that

$$\begin{aligned} & 101U_{-n} - U_{-(n-1)} \\ = \; & 101 \cdot \left(-\frac{U_n}{10^{2n}} \right) - \left(-\frac{U_{n-1}}{10^{2(n-1)}} \right) \\ = \; & -\frac{101U_n - 100U_{n-1}}{10^{2n}} \; = \; -\frac{U_{n+1}}{10^{2n}} \; ; \end{aligned}$$

as required. □

Taking into account the literature, for example in [11, 12], the function

$$f(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \tag{2.7}$$

is known as the ordinary generating function for the sequence $\{a_0, a_1, a_2, \ldots\}$.

Our next result presents the ordinary generating function for *One-Zero* sequence.

Proposition 2.5. *The ordinary generating function for the One-Zero sequence $\{U_n\}_{n\geq 0}$, denoted by $G_{U_n}(x)$, is:*

$$G_{U_n}(x)(x)=\frac{x}{1-101x+100x^2}\ .$$

Proof. According to Equation (2.7), the generating function for the *One-Zero* sequence is $G_{U_n}(x) = \sum_{n=0}^{\infty} U_n x^n$, then using the equations $-101xG_{U_n}$ and $100x^2G_{U_n}$, we obtain

$$\begin{aligned}
G_{U_n}(x) =& U_0 + U_1x + U_2x^2 + \ldots + U_nx^n + \ldots\\
-101xG_{U_n}(x) =& -101U_0x - 101U_1x^2 - 101U_2x^3 - \ldots - 101U_nx^{n+1} - \ldots\\
100x^2G_{U_n}(x) =& 100U_0x^2 + 100U_1x^3 + 100U_2x^4 + \ldots + 100U_nx^{n+2} - \ldots
\end{aligned}$$

When we add to both sides of these equations, we have

$$\begin{aligned}
& (1-101x+100x^2)G_{U_n}(x)\\
= \;& U_0 + (U_1 - 101U_0)x + (U_2 - 101U_{-1} + 100U_0)x^2 +\\
& +(U_3 - 101U_2 + 100U_1)x^3 + \ldots + (U_n - 101U_{n-1} + 100U_{n-2})x^n \ldots\ .
\end{aligned}$$

Making use of Equation (2.1), we conclude that

$$(1-101x+100x^2)G_{U_n}(x) = U_0 + (U_1 - 101U_0)x\ ,$$

since $U_0 = 0$, $U_1 = 1$ and $(1-101x+100x^2) \neq 0$, we have the result. □

Now, we express the ordinary generating function for *One-Zero* sequence with negative subscripts.

Proposition 2.6. *The ordinary generating function for the One-Zero sequence $\{U_{-n}\}_{n\geq 0}$ with negative subscripts, denoted by $G_{U_{-n}}(x)$, is:*

$$G_{U_{-n}}(x)(x)=\frac{100\ x}{100-101x+x^2}\ .$$

Proof. By Equation (2.7), the ordinary generating function for the *One-Zero* sequence is $G_{U_{-n}}(x) = \sum_{n=0}^{\infty} U_{-n}x^n$, then using the equations $\frac{-101}{10^2}xG_{U_{-n}}$ and $\frac{1}{10^2}x^2G_{U_{-n}}$, we obtain

$$\begin{aligned}
G_{U_{-n}}(x) =& U_0 + U_{-1}x + U_{-2}x^2 + \ldots + U_{-n}x^n + \ldots\\
\frac{-101}{10^2}xG_{U_{-n}}(x) =& \frac{-101}{10^2}U_0x\frac{-101}{10^2}U_{-1}x^2\frac{-101}{10^2}U_{-2}x^3 - \ldots \frac{-101}{10^2}U_{-n}x^{n+1} - \ldots\\
\frac{1}{10^2}x^2G_{U_{-n}}(x) =& \frac{1}{10^2}U_0x^2 + \frac{1}{10^2}U_{-1}x^3 + \frac{1}{10^2}U_{-2}x^4 + \ldots + \frac{1}{10^2}U_{-n}x^{n+2} - \ldots
\end{aligned}$$

When we add to both sides of these equations, we have

$$\begin{aligned}
& (1-\frac{101}{10^2}x+\frac{1}{10^2}x^2)G_{U_n}(x)\\
= \;& U_0 + (U_1 - \frac{101}{10^2}U_0)x + (U_2 - \frac{101}{10^2}U_{-1} + \frac{1}{10^2}U_0)x^2 +\\
& +(U_3 - \frac{101}{10^2}U_2 + \frac{1}{10^2}U_1)x^3 + \ldots + (U_n - \frac{101}{10^2}U_{n-1} + \frac{1}{10^2}U_{n-2})x^n \ldots\ .
\end{aligned}$$

Making use of Equation (2.6), we conclude that

$$\left(1-\frac{101}{10^2}x+\frac{1}{10^2}x^2\right)G_{U_n}(x)=U_0+(U_1-\frac{101}{10^2}U_0)x\ .$$

Since $U_0=0$, $U_1=1$ we have

$$\frac{100-101x+x^2}{10^2}G_{U_n}(x)=1x\ ,$$

with $(1-101x+100x^2)\neq 0$, and we have the result. □

In accordance with the established literature, see [12], the exponential generating function, designated as $E_{a_n}(x)$ of a sequence $\{a_n\}_{n\geq 0}$ is a power series of the form

$$E_{a_n}=a_0+a_1x+\frac{a_2x^2}{2!}+...+\frac{a_nx^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{a_nx^n}{n!}\ .$$

In the next result we consider the Binet Equation (2.3), and obtain the exponential generating function for the *One-Zero* sequence $\{U_n\}_{n\geq 0}$.

Proposition 2.7. *For all $n\geq 0$ the exponential generating function for the One-Zero sequence $\{U_n\}_{n\geq 0}$ is*

$$E_{U_n}(x)=\frac{1}{99}\left(e^{100x}-e^x\right)\ .$$

Proof. The exponential generating function for the *One-Zero* numbers is $\sum_{n=0}^{\infty}\frac{U_nt^n}{n!}$. Using Equation (2.3), we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{U_nx^n}{n!}&=\sum_{n=0}^{\infty}\frac{10^{2n}-1}{99}\cdot\frac{x^n}{n!}\\&=\frac{1}{99}\left(\sum_{n=0}^{\infty}\frac{(100x)^n}{n!}-\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\\&=\frac{1}{99}\left(e^{100x}-e^x\right)\ ,\end{aligned}$$

as required. □

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is given by:

$$P_{a_n}(x)=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}e^{-x}=e^{-x}E_{a_n}(x)\ .$$

Consequently, the corresponding Poisson generating function is derived.

Corollary 2.8. *For all $n\geq 0$ the Poisson generating function for the One-Zero sequence $\{U_n\}_{n\geq 0}$ is*

$$P_{U_n}(x)=\frac{1}{99}(e^{99x}-1)\ .$$

In a similar manner, we express the exponential generating function and Poisson generating function for *One-Zero* sequence with negative subscripts.

Proposition 2.9. *The exponential generating function for the One-Zero sequence $\{U_{-n}\}_{n\geq 0}$ with negative subscripts, denoted by $E_{U_{-n}}(x)$, is:*

$$E_{U_{-n}}(x)(x)=-\frac{1}{99}(e^x-e^{\frac{x}{100}})\ .$$

Proof. Similarly, we will treat the negatives in an analogous manner.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{U_{-n}x^n}{n!} &= \sum_{n=0}^{\infty}-\frac{10^{2n}-1}{99\cdot 10^{2n}}\cdot\frac{x^n}{n!}\\ &= -\frac{1}{99}\left(\sum_{n=0}^{\infty}\frac{100^n\cdot x^n}{100^n\cdot n!}-\sum_{n=0}^{\infty}\frac{x^n}{n!\cdot 100^n}\right)\\ &= -\frac{1}{99}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{\frac{1}{100^n}x^n}{n!}\right)\\ &= -\frac{1}{99}(e^x-e^{\frac{x}{100}})\ ,\end{aligned}$$

which verifies the result. □

Corollary 2.10. *For all $n \geq 0$ the Poisson generating function for the One-Zero sequence $\{U_{-n}\}_{n\geq 0}$ with negative subscripts, is*

$$P_{U_{-n}}(x) = -\frac{1}{99}(1-e^{\frac{-99x}{100}})\ .$$

3 Some identities for *One-Zero* sequence

In this section, we establish some classical identities for the *One-Zero* sequence for all integers n, for example, the Tagiuri-Vajda, Catalan, Cassini, d'Ocganes and Gelin-Cesàro identities.

The first result establishes the addition formula for two terms of the *One-Zero* sequence.

Proposition 3.1. *For all non-negative integers m, n , we get*

$$U_mU_{n+1} - 10^2U_{m-1}U_n = 99U_{m+n}\ .$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. According to Binet's Equation (2.3), we have

$$\begin{aligned}&U_mU_{n+1}-10^2U_{m-1}U_n\\ =\ &\left(\frac{10^{2m}-1}{99}\right)\left(\frac{10^{2(n+1)}-1}{99}\right)-10^2\left(\frac{10^{2(m-1)}-1}{99}\right)\left(\frac{10^{2n}-1}{99}\right)\\ =\ &\frac{10^{2(n+m+1)}-10^{2m}-10^{2(n+1)}-10^2\left(10^{2(n+m-1)}-10^{2(m-1)}-10^{2n}+1\right)}{99}\\ =\ &\frac{10^{2(n+m+1)}-10^{2(n+m)}-10^2+1}{99}\\ =\ &\frac{(10^{2(m+n)}-1)(10^2-1)}{99}\ =\ 99U_{m+n}\ ,\end{aligned}$$

as required. □

In consequence of Proposition 3.1, the following result is established.

Corollary 3.2. *For all non-negative integers k and let $\{U_k\}_{k\geq 0}$ be the One-Zero sequence. Then the following identities hold:*

(a) $99U_{2k+1} = U_{k+1}^2 - 10^2U_k^2$;

(b) $99U_{2k} = 101U_k^2 - 2\cdot 10^2U_kU_{k-1}$.

Proof. (a) As $2k+1 = (k+1)+k$, for all non-negative integer k. In Proposition 3.1 make $m = k+1$ and $n = k$, so it follows that

$$\begin{aligned}99U_{2k+1} &= 99U_{(k+1)+k}\\ &= U_{k+1}U_{k+1} - 10^2U_kU_k\\ &= U_{k+1}^2 - 10^2U_k^2\ .\end{aligned}$$

(b) In Proposition 3.1 make $m = k$ and $n = k$, so it follows that

$$\begin{aligned} 99U_{2k} &= U_kU_{k+1} - 10^2U_{k-1}U_k \\ &= U_k(U_{k+1} - 10^2U_{k-1}) \ . \end{aligned}$$

According to recurrence Equations (2.1), we have $U_{k+1} = 101U_k - 10^2U_{k-1}$. Then:

$$\begin{aligned} 99U_{2k} &= U_k(101U_k - 100U_{k-1} - 10^2U_{k-1}) \\ &= 101U_k^2 - 2 \cdot 10^2U_kU_{k-1} \ , \end{aligned}$$

which verifies the result. □

The following result establishes a linear combination formula for the sum of two terms. The proof of this result and the subsequent four results will not be presented in order to avoid any potential tedium for the reader. The proof is analogous to that of Proposition 3.1.

Proposition 3.3. *For all non-negative integers m, n , we get*

$$U_mU_{n+1} - 10^4U_{m-1}U_n + 1 = U_m + U_{n+1} \ ,$$

where $\{U_n\}_{n\geq 0}$ is the *One-Zero* sequence.

The following result shows the relationship between the double order term and the quadratic order term.

Proposition 3.4. *Let n be any non-negative integer. We have*

$$U_{2n} - 2U_n = 99U_n^2 \ ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero numbers.

The convolution identity is shown below, except for one constant.

Proposition 3.5. *For all non-negative integers m, n , we get*

$$99^2(U_{m-1}U_n + U_mU_{n+1}) = 1001 \cdot 10^{2(m+n-1)} - 101(10^{2m-1} + 10^{2n}) + 2 \ ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Now let us make a result that will help us further.

Proposition 3.6. *For all integer non-negative n and k, with $n \geq k$, we have*

$$99 \cdot U_{n-k}U_{n+k} \ = \ U_{2n} - (U_{n-k} + U_{n+k}) \ , \tag{3.1}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

The following result shows a difference between two products; this result will be used later.

Proposition 3.7. *Let m be any natural number. We have*

$$U_{m+3}U_{m+4} - U_{m+1}U_{m+6} = (10^4 + 1)10^{2(m+1)} \cdot U_3 \ ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

The Tagiuri-Vajda identity for the sequence *One-Zero* is given below.

Theorem 3.8. *Let n, k, r be any natural numbers. We have*

$$U_{n+k}U_{n+r} - U_nU_{n+k+r} \ = \ 10^{2n}U_rU_k \ ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. Using Equation (2.3) again we obtain that

$$
\begin{aligned}
& U_{n+k}U_{n+r} - U_nU_{n+k+r} \\
= & \left(\frac{10^{2(n+k)}-1}{99}\right)\left(\frac{10^{2(n+r)}-1}{99}\right) - \left(\frac{10^{2n}-1}{99}\right)\left(\frac{10^{2(n+k+r)}-1}{99}\right) \\
= & \frac{10^{2(2n+k+r)} - 10^{2(n+k)} - 10^{2(n+r)} + 1}{99^2} - \frac{10^{4n+2k+2r} - 10^{2n} - 10^{2(n+k+r)} + 1}{99^2} \\
= & \frac{10^{2(n+k+r)} - 10^{2(n+k)} - 10^{2(n+r)} + 10^{2n}}{99^2} \\
= & \frac{10^{2(n+k)}(10^{2r}-1) - 10^{2n}(10^{2r}-1)}{99^2} \\
= & \frac{(10^{2r}-1)(10^{2(n+k)} - 10^{2n})}{99^2} \\
= & \frac{10^{2n}(10^{2r}-1)(10^{2k}-1)}{99^2} ,
\end{aligned}
$$

and we have the validity of the result. □

The following identities will be derived as a consequence of the Tagiuri-Vajda Identity, as demonstrated in Theorem 3.8. The subsequent results will present the detailed derivation.

Corollary 3.9. *(d'Ocagne's Identity) Let m, n be any natural. For $m \geq n$ we have*

$$U_mU_{n+1} - U_{m+1}U_n \;=\; 10^{2n}U_{m-n} ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. First, we consider $k = m - n$ and $r = 1$ in Theorem 3.8, so

$$U_mU_{n+1} - U_nU_{m+1} \;=\; 10^{2n}U_1U_{m-n} .$$

Since $U_1 = 1$,and we get the result. □

Similar to Corollary 3.9 we have the Catalan Identity.

Corollary 3.10. *[Catalan's Identity] Let n, k be any natural. For $n \geq k$ we have*

$$(U_n)^2 - U_{n-k}U_{n+k} \;=\; 10^{2(n-k)} \cdot U_k^2 , \tag{3.2}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. Just take $r = -k$ in Theorem 3.8 we have

$$U_{n+k}U_{n-k} - (U_n)^2 \;=\; 10^{2n}U_k \cdot U_{-k} ,$$

By Equation (2.4), we have

$$U_{n+k}U_{n-k} - (U_n)^2 \;=\; -10^{2n}U_k \cdot \frac{U_k}{10^{2k}} ,$$

and we get the result. □

A consequence of Corollary 3.10 is the following identity.

Corollary 3.11. *Let n be any natural number. For $n \geq k$ we have*

$$(U_n)^2 - U_{n-2}U_{n+2} \;=\; 10201 \cdot 10^{2(n-2)} , \tag{3.3}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. To obtain the desired result, simply substitute $k = 2$ into Equation (3.2). Given that $U_2 = 101$, the result is anticipated. □

Cassini's identity also follows directly from the Corollary 3.10.

Corollary 3.12. *(Cassini's Identity) For all $n \geq 1$, we have*

$$(U_n)^2 - U_{n-1}U_{n+1} = 10^{2(n-1)} , \tag{3.4}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. Just make $k = 1$ in the Equation (3.2). □

By substituting $n = 2m$ in Corollary 3.12, we conclude:

Corollary 3.13. *For all $m \geq 1$, we have*

$$(U_{2m})^2 - U_{2m-1}U_{2m+1} = 10^{2(2m-1)} ,$$

where $\{U_m\}_{m\geq 0}$ is the One-Zero sequence.

Now, we present the Gelin-Cesàro identity for the *One-Zero* sequence.

Proposition 3.14. *Let n be any natural number. Then the identity holds*

$$\begin{aligned} & (U_n)^4 - U_{n-2}U_{n-1}U_{n+1}U_{n+2} \\ = \; & 10201\Big(10^{2(2n-3)} + U_{n-2}\left[U_{2n} - (U_{n-1} + U_{n+1})\right] + U_{n-1}\left[U_{2n} - (U_{n-2} + U_{n+2})\right]\Big) , \end{aligned}$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. According to Equation (3.3), we have

$$(U_n)^2 = U_{n-2}U_{n+2} + 10201 \cdot 10^{2(n-2)} , \tag{3.5}$$

By Equation (3.4)

$$(U_n)^2 = U_{n-1}U_{n+1} + 10^{2(n-1)} , \tag{3.6}$$

Multiplying both sides of the Equations (3.5) and (3.6), we get

$$\begin{aligned} (U_n)^4 \; = \; & U_{n-2}U_{n-1}U_{n+1}U_{n+2} + 10201 \cdot 10^{2(n-2)}U_{n-1}U_{n+1} \\ & +10^{2(n-1)}U_{n-2}U_{n+2} + 10201 \cdot 10^{2(2n-3)} , \end{aligned}$$

or equivalent,

$$\begin{aligned} & (U_n)^4 - U_{n-2}U_{n-1}U_{n+1}U_{n+2} \\ = \; & 10201 \cdot 10^{2(2n-3)} + 10201 \cdot U_{n-2}\left[U_{2n} - (U_{n-1} + U_{n+1})\right] + U_{n-1}\left[U_{2n} - (U_{n-2} + U_{n+2})\right] , \end{aligned}$$

this completes the proof. □

Finally, we present an interesting result that investigates combinations of certain terms within the *One-Zero* sequence. This result shares notable parallels with the Tagiuri-Vajda, Catalan, Cassini, and d'Ocagne identities discussed earlier.

Proposition 3.15. *Let m be any natural. We have*

$$(U_{m+3})^3 - U_{m+1}U_{m+2}U_{m+6} = 10^{2(m+1)}[10^2U_{m+3} + (10^4 + 1)U_3U_{m+2}] ,$$

where $\{U_n\}_{n\geq 0}$ is the One-Zero sequence.

Proof. By Cassini's Identity, Corollary 3.12, we have $(U_{m+3})^2 - U_{m+2}U_{m+4} = 10^{2(m+2)}$. So $(U_{m+3})^3 - U_{m+2}U_{m+3}U_{m+4} = 10^{2(m+2)}U_{m+3}$. According to Proposition 3.7, we get

$$\begin{aligned} (U_{m+3})^3 - U_{m+2}U_{m+3}U_{m+4} &= 10^{2(m+2)}U_{m+3} \\ (U_{m+3})^3 - U_{m+2}(U_{m+1}U_{m+6} + (10^4+1)10^{2(m+1)} \cdot U_3) &= 10^{2(m+2)}U_{m+3} \ , \end{aligned}$$

or equivalently

$$\begin{aligned} & (U_{m+3})^3 - U_{m+1}U_{m+2}U_{m+6} \\ = \ & 10^{2(m+2)}U_{m+3} + (10^4+1)10^{2(m+1)}U_3U_{m+2} \\ = \ & 10^{2(m+1)}[10^2U_{m+3} + (10^4+1)U_3U_{m+2}] \ , \end{aligned}$$

the end of the proof. □

To finish this section, the following result presents another interesting application of Cassini's Identity, as shown below.

Proposition 3.16. *For any positive integer m, the Diophantine equation $x^2 - 101xy + 100y^2 = 10^{2(m-1)}$ has infinitely many solutions.*

Proof. By combining Equation (3.2) with Equation (2.1), we obtain the following equation: $(U_m)^2 - 101U_{m-1}U_m + 100(U_{m-1})^2 = 10^{2(m-1)}$. Setting $U_m = x$ and $U_{m-1} = y$ yields the desired result, where $\{U_n\}_{n\geq 0}$ is the *One-Zero* sequence. □

4 Some Sum formulas

In this section, we present results on partial sums of the terms of the *One-Zero* sequence with n integers. Initially, consider the sequence of partial sums $\sum_{k=0}^{n} U_k = U_0 + U_1 + U_2 + \cdots + U_n$, for $n \geq 0$, where $\{U_n\}_{n\geq 0}$ is the *One-Zero* sequence.

We will present two results involving the partial sum of the terms of the *One-Zero* sequence.

Proposition 4.1. *Let $\{U_n\}_{n\geq 1}$ be the One-Zero sequence and n the non-negative integer, then:*

$$\begin{aligned} (a) \quad \textstyle\sum_{k=0}^{n} U_k &= \frac{U_{n+2} - (n+1)}{99} \ , \\ (b) \quad \textstyle\sum_{k=0}^{n} U_{2k} &= \frac{U_{n+1}[10^{2(n+1)} + 1]}{999} - \frac{(n+1)}{99} \ , \\ (c) \quad \textstyle\sum_{k=0}^{n} U_{2k+1} &= \frac{10^2U_{n+1}(10^{2(n+1)} + 1)}{999} - \frac{(n+1)}{99} \ . \end{aligned}$$

Proof. (a) By Equation (2.3) we have,

$$\begin{aligned} \sum_{k=0}^{n} U_k &= U_0 + U_1 + U_2 + \cdots + U_n \\ &= \frac{10^0 - 1}{99} + \frac{10^2 - 1}{99} + \frac{10^4 - 1}{99} + \cdots + \frac{10^{2n} - 1}{99} \\ &= \frac{10^0 + 10^2 + \cdots + 10^{2n} - (1 + 1 \cdots + 1)}{99} \\ &= \frac{10^0\left(\frac{10^{2(n+1)}-1}{10^2-1}\right) - (n+1)}{99} \\ &= \frac{U_{n+2} - (n+1)}{99} \ , \end{aligned}$$

as required.

(b) Note that, Equation (2.3) once more, we get

$$\begin{aligned}
\sum_{k=0}^{2n} U_k &= U_0 + U_2 + U_4 + \cdots + U_{2n} \\
&= \frac{10^0 - 1}{99} + \frac{10^4 - 1}{99} + \frac{10^8 - 1}{99} + \cdots + \frac{10^{4n} - 1}{99} \\
&= \frac{10^0 + 10^4 + \cdots + 10^{4n} - (1 + 1 \cdots + 1)}{99} \\
&= \frac{\left(\frac{10^{4(n+1)}-1}{10^4-1}\right) - (n+1)}{99} = \frac{\frac{[10^{2(n+1)}-1][10^{2(n+1)}+1]}{10^4-1} - (n+1)}{99} \\
&= \frac{[10^{2(n+1)} - 1][10^{2(n+1)} + 1] - 999(n+1)}{99 \cdot 999},
\end{aligned}$$

as required.

(c) Similarly, we have

$$\begin{aligned}
\sum_{k=0}^{n} U_{2k+1} &= U_1 + U_3 + \cdots + U_{2n+1} \\
&= \frac{10^2 - 1}{99} + \frac{10^6 - 1}{99} + \cdots + \frac{10^{2(2n+1)}}{99} \\
&= \frac{10^2 + 10^6 + \cdots + 10^{2(2n+1)} - (n+1)}{99} \\
&= \frac{10^2\left(\frac{10^{4(n+1)}-1}{10^4-1}\right) - (n+1)}{99} \\
&= 10^2 \frac{(10^{2(n+1)} - 1)(10^{2(n+1)} + 1) - 999(n+1)}{99 \cdot 999}
\end{aligned}$$

this completes the proving. □

Consider now the sequence of alternating partial sums given by

$$\sum_{k=0}^{n} (-1)^k U_k = U_0 - U_1 + U_2 - U_3 + \cdots + (-1)^n U_n$$

for $n \geq 0$, where $\{U_n\}_{n\geq 0}$ denotes the *One-Zero* sequence.

Proposition 4.2. *Let $\{U_n\}_{n\geq 0}$ be the One-Zero sequence, and n be a non-negative integer. Then:*

(a) $\displaystyle\sum_{k=0}^{n} (-1)^k U_k = -\frac{11U_{n+1}[10^{2(n+1)}+1]}{111},$ *if n is odd ;*

(b) $\displaystyle\sum_{k=0}^{n} (-1)^k U_k = \frac{U_{n+2}[10^{2(n+2)}+1] - 10^2 U_{n+1}(10^{2(n+1)}+1)}{999} - \frac{1}{99},$ *if n is even .*

Proof. (a) First, suppose that n is the odd natural number, or equivalently, the last term is negative, thus

$$\begin{aligned}
\sum_{k=0}^{2n+1} (-1)^k U_k &= U_0 - U_1 + U_2 - U_3 + \cdots + U_{2n} - U_{2n+1} \\
&= (U_0 + U_2 + \cdots + U_{2n}) - (U_1 + U_3 + \cdots + U_{2n+1}) \\
&= \sum_{k=0}^{n} U_{2k} - \sum_{k=0}^{n} U_{2k+1}
\end{aligned}$$

According to the Proposition 4.1, items (b) and (c), it follows that

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k U_k &= \sum_{k=0}^{n} U_{2k} - \sum_{k=0}^{n} U_{2k+1} \\ &= \frac{U_{n+1}[10^{2(n+1)}+1]}{999} - \frac{(n+1)}{99} - \left(\frac{10^2 U_{n+1}(10^{2(n+1)}+1)}{999} - \frac{(n+1)}{99}\right) \\ &= \frac{U_{n+1}[10^{2(n+1)}+1](1-10^2)}{999}\,.\end{aligned}$$

(b) In this case, we consider n even natural number, so

$$\begin{aligned}\sum_{k=0}^{2(n+1)}(-1)^k U_k &= U_0 - U_1 + U_2 - U_3 + \cdots + U_{2n} - U_{2n+1} + U_{2(n+1)} \\ &= (U_0 + U_2 + \cdots + U_{2n} + U_{2(n+1)}) - (U_1 + U_3 + \cdots + U_{2n+1}) \\ &= \sum_{k=0}^{n+1} U_{2k} - \sum_{k=0}^{n} U_{2k+1}\,.\end{aligned}$$

As in item (a), apply the Proposition 4.1. □

5 Final Considerations

In the present study, we discussed the properties of the sequence *One-Zero* with integer indexes, which is a sequence of the type Horadam $\{h_n\}_{n\geq 0}$. Our aim was to determine some identities for this specific sequence, in particular the classical identities such as Tagiuri-Vajda, D'Ocagene, Catalan and Cassini. By highlighting these results, we wish to inspire further research into this class of numbers. In addition, some aspects of these investigations appear to be pioneering, suggesting that although the results are obtained using elementary mathematical methods, they may offer original perspectives that potentially enrich the understanding of this domain. In future works we will extend and generalize this sequence to the domain of complex numbers, quaternions, octonions and hybrid numbers. Finally, we observed the similarity of the *One-Zero* sequence with the repunit sequence, not only because they are both type-Horadam sequences, as can be seen in the papers [13–15], something we should also investigate in future work.

Acknowledgments: The first author was partially supported by PROPESQ-UFT. The third author is member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT - Centre of Mathematics of University of Minho) and she thanks the Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UIDB/00013/2020 and UIDP/00013/2020.

Conflict of interest statement: The authors declare no conflict of interest in the writing of the manuscript, or in the decision to publish the results.

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