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Traveling waves in a Kermack-McKendric epidemic model

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Abstract: This study explores the existence of traveling wave solutions in the classical Kermack-McKendrick epidemic model with local diffusive. The findings highlight the critical role of the basic reproduction number R_0 in shaping wave dynamics. Traveling wave solutions are shown to exist for wave speeds $c \geq c^*$ when $R_0 > 1$, with c^* denoting the minimal wave speed. Conversely, no traveling waves are observed for $c < c^*$ or $R_0 < 1$. Numerical simulations are employed to validate the theoretical results, demonstrating the presence of traveling waves for a range of nonlinear incidence functions and offering insights into the spatial spread.

Keywords: Kermack-McKendrick model; minimal wave speed; traveling waves; basic reproduction number.

Classification MSC: 35K57; 37N25; 92D30.

1 Introduction

The Kermack-McKendrick model of an epidemic has been the cornerstone in mathematical epidemiology [1], supplying the basic insights into the dynamics of infectious diseases. While originally focused on the study of temporal development, such models have, in recent times, extended to spatial aspects, one of which involves traveling wave solutions. Traveling waves are associated with the spatial propagation of infection within a population, giving a framework for how diseases spread geographically over time [2–7].

In this context, most epidemic models consider the spatial dynamics of either individuals or infectious agents through the use of appropriate diffusion terms. In recent years, there has been a growing interest in local diffusion as a more realistic approach to modeling spatial dispersal in either heterogeneous or long-range movements [8–10].

The most fundamental question for these types of spatial epidemic models concerns the existence and properties of traveling wave solutions, which crucially depend on the basic reproduction number R_0. For $R_0 > 1$, traveling waves are normally anticipated, provided the wave speed is greater than a critical value [11–17]. On the other hand, when $R_0 \leq 0$, extinction of the disease is expected, and no traveling waves exist. Despite much activity, the interaction between R_0 wave speed, and incidence functions within the Kermack-McKendrick framework remains a topic of ongoing interest.

This paper considers the existence of traveling wave solutions in a local diffusion extension of the Kermack-McKendrick epidemic model with two cases $d_1 = 0$ and $d_2 = 0$. Through examining the threshold dynamics of the basic reproduction number and minimal wave speed c^*, we establish a rigorous criterion for wave existence. Numerical simulations are used to illustrate the theoretical findings, and examples of traveling waves with different linear incidence functions are shown[27–29]. These results deepen our insight into spatial dynamics of diseases and will be useful for controlling infectious diseases

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in non homogeneous environments see also [18–20]. Therefore, we consider the following local diffusion Kermack-McKendrick epidemic model

$$\begin{cases} \frac{\partial S}{\partial t} &= d_1\Delta S - \beta S(x,t)I(x,t) \\ \frac{\partial I}{\partial t} &= d_2\Delta I + \beta S(x,t)I(x,t) - \gamma I(x,t) \end{cases} \tag{1.1}$$

with $t > 0$, $x \in \mathbb{R}$.

2 Traveling waves for Kermack-McKendric model with $d_1 = 0$

Now, in this section we consider the model (1.1) and assuming that the diffuse rate of the susceptible individuals equal 0 ($d_1 = 0$) [21, 22]. We work to demonstrate the existence of traveling waves solutions of the following reaction diffusion system

$$\begin{cases} \frac{\partial S}{\partial t} &= -\beta S(x,t)I(x,t) \\ \frac{\partial I}{\partial t} &= d_2\Delta I + \beta S(x,t)I(x,t) - \gamma I(x,t). \end{cases}$$

With $t > 0$, $x \in \mathbb{R}$. Ignoring the diffusion term, this is just the classical Kermack-McKendric model [1]. Then, we use the following change of variable.

$$U = \frac{I}{S_0} \ \ , \ V = \frac{S}{S_0} \ \ , \ r = \frac{\gamma}{\beta S_0} \ \ , \ t^* = \beta S_0 t \ \ , \ x^* = \sqrt{(\frac{\beta S_0}{d_2})}x,$$

and if we assume there is initially a uniform distribution of susceptible, S_0. We obtian the new following system

$$\begin{cases} V_t &= -UV, \\ U_t &= \Delta U + UV - rU, \end{cases} \tag{2.1}$$

on $\mathbb{R}^n$ with initial conditions

$$U(x,0) \geq 0 \qquad V(x,0) = 1. \tag{2.2}$$

Interpreting U as density of infectives and V as density of susceptibles, these equations model a directly transmitted disease for which the spatial spread is due to diffusion of infectives. The parameter $R_0 = \frac{1}{r} > 0$ is usually referred to as the basic reproduction namber of the disease.
As a preliminary step we note that T is an invariant rectangle for (2.1) which gives the next lemma (see e.g.[23]).

$$T = \{(V,U) \in \mathbb{R}^n \ \ U \geq 0 \ \text{ and } \ 0 \leq V \leq 1\}.$$

Lemma 2.1. *Under the assumptions (2.2) the solution of (2.1) satisfies*

$$U(x,t) \geq 0 \ \ , \ \ 0 < V(x,t) \leq 1.$$

Proof. By the first equation of (2.1) we have $V(t,x)$ is decreasing and after (2.2) we get $V(x,0) = 1$, hence, we deduce that $0 < V(x,t) \leq 1$ and it is clear that the density of infectives $U(x,t) \geq 0$. □

Theorem 2.2. *If $r \geq 1$ and $U(x,0)$ has bounded support, then $U(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on $\mathbb{R}^n$. If $r > 1$ it does so exponentially in t.*

Proof.

Lemma 2.3. *Let $\phi(x) = U(x,0)$ and define*

$$U_1(x,t) = \exp^{(1-r)t}(\phi.K_t)(x),$$

where

$$K_t(x) = (4\pi t)^{\frac{-n}{2}} \exp^{\left(\frac{x}{2\sqrt{t}}\right)^2},$$

is the fundamental solution of the heat equation in $\mathbb{R}^n$. Then

$$U(x,t) \leq U_1(x,t).$$

This is (as was lemma 2.1) a consequence of the strong maximum principle for parabolic equations (see [24]). In fact, U_1 is the solution of

$$U_t - \Delta U - UV + rU \leq U_t - \Delta U + (r-1)U$$

where $0 < V(x,t) \leq 1$

$$\begin{cases} LU = U_t - \Delta U + (r-1)U = 0 \\ U(x,0) = \phi(x) \end{cases}$$

and $U(x,t)$ has the same initial condition and satisfies, if we replace

$$U_t = \Delta U + UV - rU,$$

hence

$$LU = \Delta U + UV - rU - \Delta U + (r-1)U$$

according to the principle of comparison, we get

$$LU = U(V-1) \leq 0.$$

Remarks 2.4. *If $r \geq 1$ and ϕ has compact support, then*

$$U(x,t) \leq C \exp^{(1-r)t} t^{\frac{-n}{2}},$$

where C is some constant independent of $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$.

□

2.1 Existence of Travelling waves solutions

If $r < 1$ the situation is different. We assume from now on that the space dimension equals 1. A travelling wave solution of (2.1) is a solution of the form

$$U(x,t) = f(x-ct), V(x,t) = g(x-ct),$$

where f and g are positive bounded functions of a single variable. There is no loss in generality in assuming that $c > 0$, which we do. We assume that the boundary conditions satisfiy ,

$$f(-\infty) = f(\infty) = 0, \quad g(\infty) = 1.$$

The existence of such travelling waves is given in the next theorem.

Theorem 2.5. *There exist a travelling wave solution of (2.1) (with boundary conditions), if and only if $r < 1$. Then there is a unique such wave for every $c \geq c_0 = 2\sqrt{1-r}$. We have that $a = g(-\infty)$ satisfies*

$$a - r \ln a = 1,$$

and also that

$$f_{max} < r \ln r - r + 1.$$

Proof. The travelling wave solution of (2.1) consists of functions f and g of $z = x - ct$ such that

$$\begin{cases} g' = \frac{fg}{c}, \\ \\ f'' + cf' + fg - rf = 0, \end{cases} \tag{2.3}$$

with

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} = f'(z) \Rightarrow \frac{\partial^2 f}{\partial x^2} = f''(z) \quad , \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = -c\frac{\partial f}{\partial z} = -cf'(z)$$

and

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial z}\frac{\partial z}{\partial t} = -c\frac{\partial g}{\partial z} = -cg'(z)$$

together with the boundary conditions

$$f(-\infty) = f(\infty) = 0, g(\infty) = 1,$$

and which are bounded and positive in $(-\infty, \infty)$. By assumption $c > 0$.

The rest of this proof will be devoted to the main assertion of theorem 2.5,

$$f'' + cf' + fg - rf = 0$$

divide it on c we get

$$\frac{f''}{c} + f' + \frac{fg}{c} - \frac{rf}{c} = 0 \tag{2.4}$$

by the first equation of system (2.3) we have $g' = \frac{fg}{c}$ which means that $f = \frac{cg'}{g}$ we substitute the result in (2.4)

$$\frac{f''}{c} + f' + g' - r\frac{g'}{g} = 0.$$

Integrating the two sides of the previons equation we obtain,

$$\frac{f'}{c} + f + g - r\ln(g) = constant = S(f', f, g), \tag{2.5}$$

then (2.5) implies that

$$\frac{d}{dz}S(f', f, g) = 0$$

so that

$$S(f', f, g)(z) = S(f', f, g)(\infty) = S(0, 0, 1) = 1.$$

It follows that (2.3) and the boundary conditions are equivalent to the following first order system

$$\begin{cases} g' = \frac{fg}{c}, \\ f' = c(1 + r\ln(g) - f - g), \end{cases} \tag{2.6}$$

with boundary conditions $f(\infty) = 0, g(\infty) = 1$.

2.1.1 Equilibria

As stated in the previous section.The system (2.6) admits a unique equilibrium. The following lemma shows the obtained results.

Theorem 2.6. *The system* (2.6) *admits a unique positive equilibrium* $E^* = (g^*, f^*)$ *if* $r \geq 1$ *and two positive equilibrium* $E_1 = (g_1^*, f_1^*)$ *,*$E_2 = (g_2^*, f_2^*)$ *if* $r < 1$.

Proof. The equilibrium points of the reduced system satisfies $g' = f' = 0$,we put $h(g) = 1 + r\ln(g) - g$, then we have:

$$\begin{cases} 0 = \frac{f^* g^*}{c}, \\ 0 = c(1 + r\ln(g^*) - f^* - g^*), \end{cases} \tag{2.7}$$

by the first equation of the system (2.7) we get $f^* = 0$, we substitute this value into the second equation, we have two cases:

Case 1: $r \geq 1$ then

$$0 = c(1 + r\ln(g^*) - g^*) \Rightarrow h(g^*) = 0 \Rightarrow g^* = 1,$$

we get a unique equilibrium $E^* = (1, 0)$

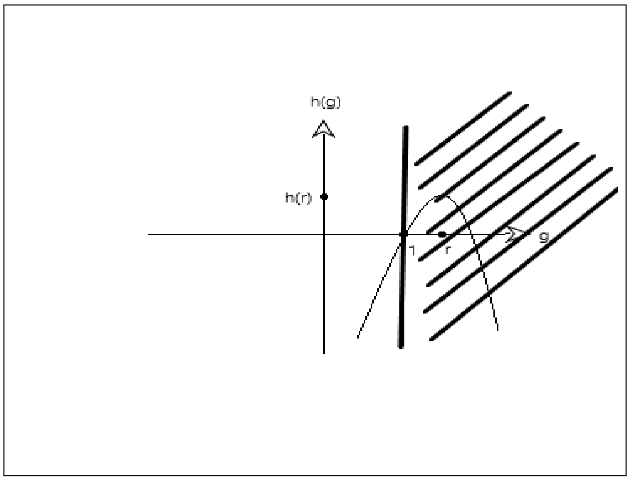

Fig. 1. The existence of equlibria for $r \geq 1$.

Case 2: $0 < r < 1$, therefore $0 = c(1 + r\ln(g^*) - g^*)$ hence $h(g^*) = 0$ which give $g^* = 1$ or $g^* = a$ / with a is a positive constant satisfies $0 < a < r < 1$, Then, there is two equilibriums $E_1 = (1, 0)$ and $E_2 = (a, 0)$

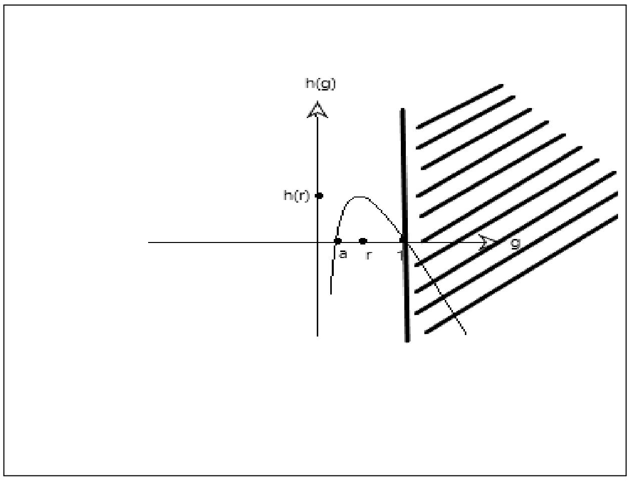

Fig. 2. The existence of equlibria for $0 < a < r < 1$.

□

2.1.2 Stability

Theorem 2.7. *The equilibrium E_1 is globally asymptotically stable if $r < 1$, $c \geq c_0$ and E_2 is unstable .*

Proof. Let $E = (g, f)$ be an arbitrary equilibrium, then, the Jacobian matrix of the system (2.6) at E

is given by

$$J(g,f) = \begin{pmatrix} \frac{f}{c} & \frac{g}{c} \\ c(\frac{r}{g}-1) & -c \end{pmatrix}.$$

Now we show the stability of the equilibrium E_1.

$$J(1,0) = \begin{pmatrix} 0 & \frac{1}{c} \\ c(r-1) & -c \end{pmatrix}.$$

The corresponding characteristic polynomial is given by

$$P(\lambda) = \lambda^2 + c\lambda - r + 1 = 0.$$

The discriminant of the equation is

$$\delta = c^2 - 4(1-r),$$

where $r < 1$, we have $\delta > 0$ and $P(\lambda)$ admet two roots

$$\lambda_1 = \frac{1}{2}\Big(-c + \sqrt{c^2 - 4(1-r)}\Big) \quad , \quad \lambda_2 = \frac{1}{2}\Big(-c - \sqrt{c^2 - 4(1-r)}\Big),$$

then,one of the following cases ,

1) If $c^2 < 4(1-r)$, we have complex eigenvalues and all of the trajectory cannot stay in the positive quadrant near E_1.
2) If $c \geq c_0 = 2\sqrt{(1-r)}$, so $P(\lambda)$ admet two negative rootes

$$\lambda_{1,2} = \frac{1}{2}\Big(-c \pm \sqrt{c^2 - 4(1-r)}\Big) < 0.$$

Clearly, E_1 is stable.

Now we move to the second equilibrium E_2,if $r > a$ then $\delta > 0$ and $P(\lambda)$ admet two rootes

$$\lambda_1 = \frac{1}{2}\Big(-c + \sqrt{c^2 - 4(a-r)}\Big) > 0 \quad , \quad \lambda_1 = \frac{1}{2}\Big(-c - \sqrt{c^2 - 4(a-r)}\Big) < 0,$$

morever, E_2 is saddle point with the unstable trajectory. We want to show that this trajectory enters $E^*(1,0)$ as $z \longrightarrow \infty$ (we proofed the globale stability of E_1). For this, let

$$D = \Big\{(g,f)\ ;\ 0 < f < mh(g)\ ,\ a < g < 1\Big\}$$

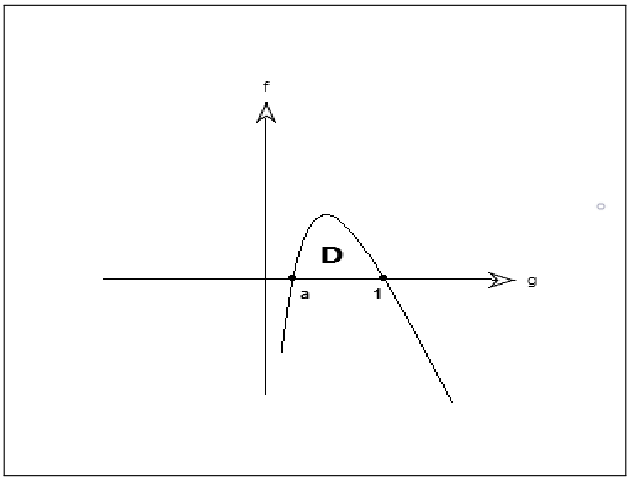

Fig. 3. The boundary and attractive domain D .

We wish to demonstrate that we may select m in such a way that the trajectory on any non stationary point on D's border enters D. Since $h(g)>0$ for $a<g<1$, we really have that $(g',f')=(0,ch(g))$ on $f=0$, which leads towards D.The equation of the tangent may be used to get the inward normal, n, of the curve $f=mh(g)$.

$$f = mh'(g_0)(g-g_0)+mh(g_0)\ ;\ \forall g_0$$

hence

$$-f+mh'(g_0)g-mh'(g_0)g_0+mh(g_0)=0\ ;\ \forall g_0$$

written from the shap

$$ax+by+c=0$$

with $n=(a,b)$ is the normal vectore. Finally,

$$n=(mh'(g),-1)=-(-mh'(g),1)=-\big(m\big(1-\frac{r}{g}\big),1\big)$$

and it is easily checked that

$$\begin{aligned}(g',f').n &= -\tfrac{fg}{c}\big(m\big(1-\tfrac{r}{g}\big)\big)-c(1+rln(g)-f-g)\\ &= -\tfrac{mf}{c}(g-r)-c(1+r\ln(g)-g)+cf\\ &= -\tfrac{m^2h(g)}{c}(g-r)-ch(g)+cmh(g)\\ &= -\tfrac{h(g)}{c}\Big[m^2(g-r)-c^2(m-1)\Big],\end{aligned}$$

with $f=mh(g)$. Which is negative if the term in the bracket is positive and that in turn is true for $a<g<1$ if it is true for $g=1$. But then the term in the bracket is

$$\frac{m^2(g-r)-c^2(m-1)}{1-r}=\left(m-\frac{c^2}{2(1-r)}\right)^2+\left(1-\frac{c^2}{4(1-r)}\right)\frac{c^2}{1-r}$$

such that we can choose $m=\frac{c^2}{2(1-r)}\geq 2$ if $c^2\geq 4(1-r)$. The unstable trajectory from $E_2=(a,0)$ reaches D, as $f'>0$ at the beginning of it. This means that $f<mh(g)$ there, with $m=1$, according to (2.6).In conclusion, the trajectory from $E_2=(a,0)$ must proceed to $E_1=(1,0)$ since it enters D and cannot exit it in a limited amount of time.In addition to $f_{max}<r\ln r-r+1$, which is shown by the fact that $f'=0$ if and only if $f=h(g)\leq h(r)$ (that we cannot have equality derives from (2.3)), this proves theorem 2.5. At last, we have demonstrated that $E_1=(1,0)$ is globally stable.

Remarks 2.8. *If $r<1$ and $c>c_0$, then the equilibrum E_1 of the system (2.6) is globally asymptotically stable then there is travelling waves solutions.*

□

Theorem 2.9. *The equilibrium E^* of the system (2.6) is unstable if $r\geq 1$.*

Proof.

$$J(1,0)=\begin{pmatrix}0 & \frac{1}{c}\\ c(r-1) & -c\end{pmatrix}.$$

The corresponding characteristic polynomial is given by

$$P(\lambda)=\lambda^2+c\lambda-r+1=0.$$

The discriminant of the equation is

$$\delta=c^2-4(1-r)>0$$

where $r\geq 1$,we have $P(\lambda)$ admet two roots

$$\lambda_1=\frac{1}{2}\Big(-c+\sqrt{c^2-4(1-r)}\Big)>0\ \ ,\ \ \lambda_1=\frac{1}{2}\Big(-c-\sqrt{c^2-4(1-r)}\Big)<0,$$

then, E^* is instable, hence no travelling wave exists . Next, we have the following results □

Remarks 2.10. *In $r \geq 1$ then the unique equlibrume E^* is unstable, which means that there is no traveling wave solution.*

□

2.2 The Propagation of Disturbances

Any starting function $U(x, 0)$ with limited support divides into two traveling waves traveling in opposing directions with the same speed, according to numerical simulations. This speed is determined by the following theorem.

Theorem 2.11. *Let $U(x, 0)$ have bounded support. Then to every $\xi > 0$, there exist N such that, for every $t > 0$,*

$$U(x + c_0 t - \frac{\ln(t)}{c_0}, t) \leq \xi \qquad \text{for} \quad x > N.$$

The interpretation of this theorem is as follows [25]. If you travel with speed

$$c(t) = \frac{d(x + c_0 t - \frac{\ln(t)}{c_0})}{dt} = c_0 - \frac{1}{c_0 t}$$

you will never be overtaken by the virus as you go towards $+\infty$, beginning to the right of the support of $U(x, 0)$. As a result, the infection's asymptotic speed must be smaller than $c(t)$. Therefore, for the one with the lowest speed c_0, $U(x, t)$ must take the shape of a travelling wave if it does for big t.

Proof. For $n = 1$ the function U_1 in lemma 2.3 is given by

$$U_1(x, t) = (4\pi t)^{\frac{-1}{2}} \exp^{(1-r)t} \int_{-\infty}^{+\infty} \exp^{\frac{-(x-y)^2}{4t}} \phi(y) dy$$

if we remplace x by $x + ct + s\ln(t)$ we get

$$U_1(x + ct + s\ln(t), t) = (4\pi t)^{\frac{-1}{2}} \exp^{(1-r)t} \int_{-\infty}^{+\infty} \exp^{\frac{-(-y+x+ct+sln(t))^2}{4t}} \phi(y) dy$$

hence

$$(x - y + ct + s\ln(t))^2 = c^2 t^2 + (-y + x + sln(t))^2 + 2(-y + x + sln(t))ct$$

it follows that

$$U_1(x + ct + s\ln(t), t) = \alpha(t) \int_{-\infty}^{+\infty} \exp^{\frac{-(x+ct+s\ln(t)-y)^2}{4t} + \frac{cy}{2}} \phi(y) dy$$

where

$$\alpha(t) = (4\pi t)^{\frac{-1}{2}} \exp^{(1-r)t} \exp^{\frac{-c^2t^2-2cts\ln(t)-2cx}{4t}} = (4\pi t)^{\frac{-1}{2}} \exp^{\frac{-(c^2-4(1-r))t-2cs\ln(t)-2cx}{4}}$$

If we take $s = -\frac{1}{c}$ it follows that if

$$\int_{-\infty}^{+\infty} \exp^{\frac{-(-y+x+ct-\frac{1}{c}\ln(t))^2}{4t} + \frac{cy}{2}} \phi(y) dy \leq \int_{-\infty}^{+\infty} \exp^{\frac{cy}{2}} |\phi(y)| dy = L < \infty$$

then

$$\begin{aligned} U_1(x + ct - \tfrac{1}{c}ln(t), t) &\leq (4\pi t)^{\frac{-1}{2}} \exp^{\frac{-(c^2-4(1-r))t-2cx}{4}} \exp^{\frac{1}{2}\ln(t)} L \\ &= (4\pi t)^{\frac{-1}{2}} \exp^{\frac{-(c^2-4(1-r))t-2cx}{4}} \exp^{\ln(\sqrt{t})} L \\ &= (4\pi t)^{\frac{-1}{2}} \exp^{\frac{-(c^2-4(1-r))t-2cx}{4}} t^{\frac{1}{2}} L \\ &= (4\pi)^{\frac{-1}{2}} \exp^{\frac{-(c^2-4(1-r))t-2cx}{4}} L \end{aligned}$$

If $c^2 \geq 4(1 - r)$ it follows, using lemma 2.3, that

$$U_l(x + ct - \frac{1}{c}\ln(t), t) \leq C \exp^{\frac{-cx}{2}}$$

where C is independent of t. Taking $c = c_0$ this proves Theorem 2.11. □

2.3 Numerical Simulation

In this section, we carry out the numerical simulations of the proposed model to illustrate our theoretical results.The global dynamics of the system (2) in two cases $r < 1$ and $r > 1$, for the values:$d_2 = 0.1$, $\gamma = 0.1$ and $\beta = 0.18$, and the initial conditions $S(0, x) = 7$ and $I(0, x) = 3$,

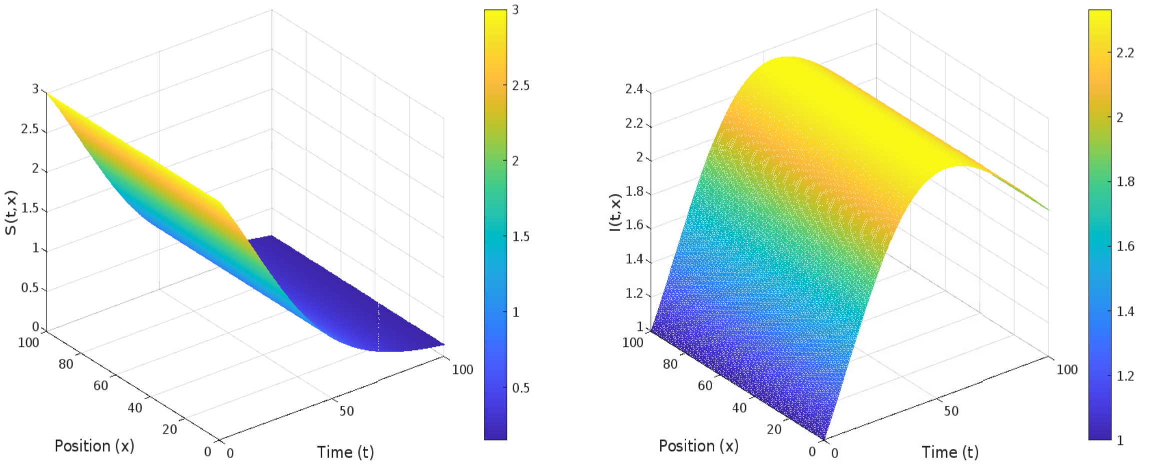

Fig. 4. Numerical simulations of solutions for system (2) where $r = 0.66 < 1$.

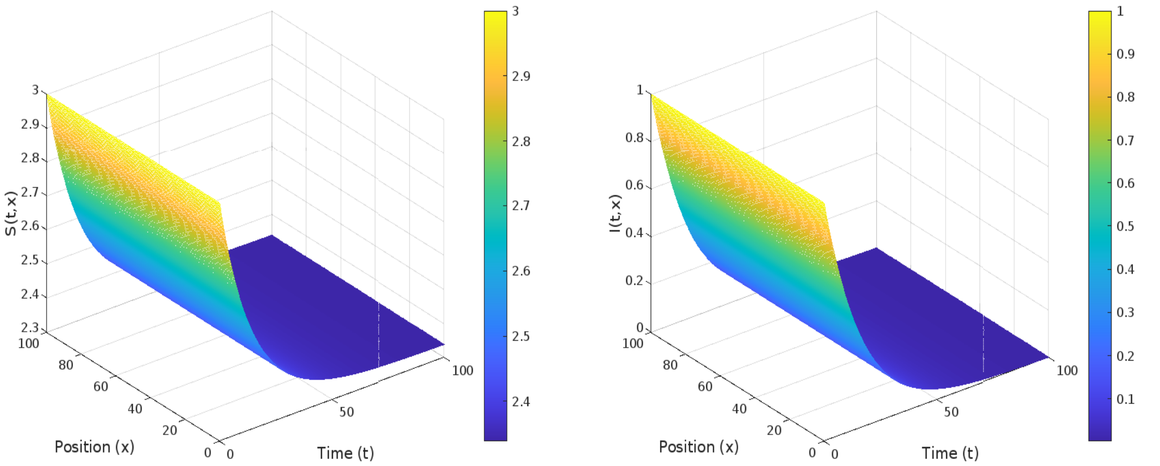

Fig. 5. Numerical simulations of solutions for system (2) where $r = 6.66 > 1$.

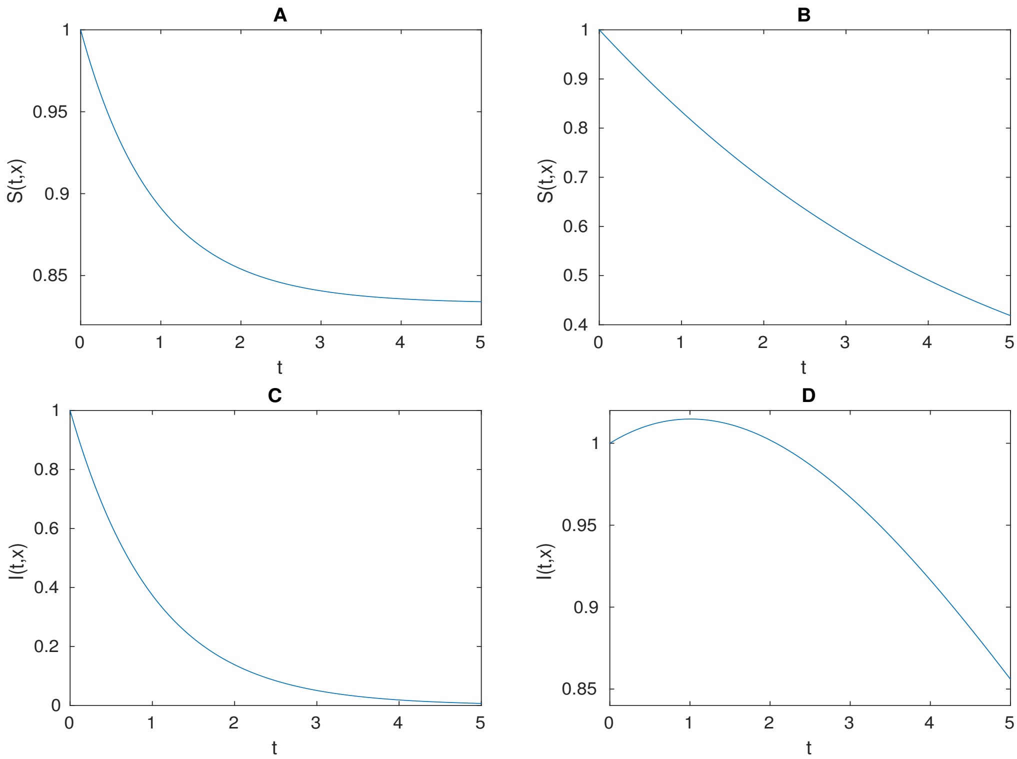

Fig. 6. The global dynamics of (2) compared to time (t),where for the left hand figure (A and C), we consider $r = 1.66 >$ 1, and for the right hand figure(B and D), we let $r = 0.55$<1.

To confirm the existence of traveling wave solutions of system (2). We take the parameters of the model as below

$$\beta = 0.5, \quad \gamma = 0.3, \quad d_2 = 0.1, \quad S(x, 0) = 4, \quad I(x, 0) = 2.$$

Using these parameters, we obtain the basic reproduction number $R_0 = 1.6667 > 1$, the minimal speed $c_0 = 1.2649$ and $r = 0.6$

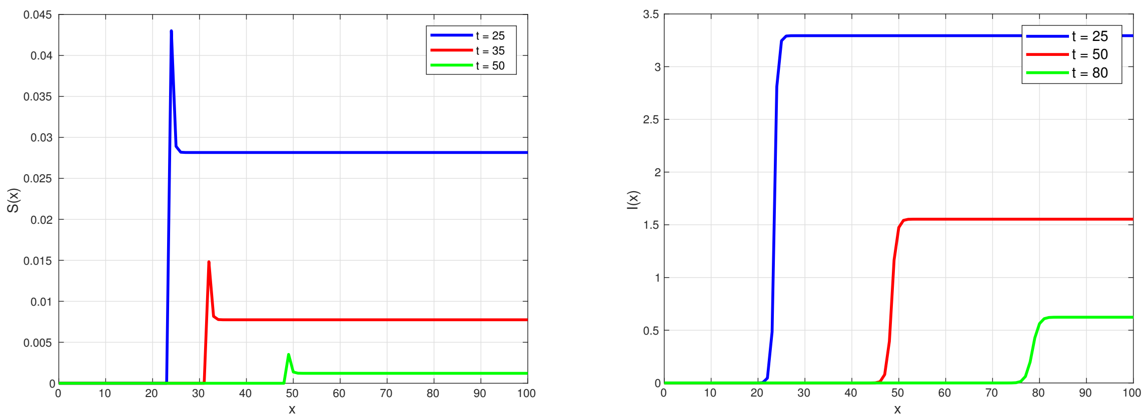

Fig. 7. The traveling wave observed in system (2) .

3 Travelling waves for Kermack-McKendric model with $d_2 = 0$

In this section we have the following reaction diffusion system

$$\begin{cases} \frac{\partial S}{\partial t} &= d_1 \Delta S - \beta S(x,t)I(x,t) \\ \frac{\partial I}{\partial t} &= \beta S(x,t)I(x,t) - \mu I(x,t) \end{cases} \tag{3.1}$$

with Ω is a bounded domain in $\mathbb{R}^n$, hence:

$$\begin{cases} \frac{\partial S}{\partial x} = \frac{\partial I}{\partial x} = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty) \\ S(x,0) = S_0(x) \geq 0 \quad , \quad I(x,0) = I_0(x) \geq 0 \quad \text{in} \quad \Omega. \end{cases}$$

By the same changement of variable, that is,

$$U = \frac{I}{S_0} \quad , \ V = \frac{S}{S_0} \quad , \ r = \frac{\mu}{\beta S_0} \quad , \ t^* = \beta S_0 t \quad , \ x^* = \sqrt{(\frac{\beta S_0}{D})}x,$$

we get

$$\begin{cases} V_t &= \Delta V - UV \quad x \in \mathbb{R}, \\ U_t &= UV - rU \quad x \in \mathbb{R}, \end{cases} \tag{3.2}$$

in $\mathbb{R}^n$ with initial conditions

$$U(x,0) \geq 0 \qquad V(x,0) = 1.$$

3.1 Establish the existence of traveling waves solutions

In this section, we study the existence of traveling wave solutions of system (3.2).

Theorem 3.1. *Can't tell if* (3.3) *admit a traveling waves solutions if* $r < 1$.

Proof. A travelling wave solution of 3.2 is a solution of the form

$$U(x,t) = f(x - ct), V(x,t) = g(x - ct)$$

. Where f and g are positive bounded functions of a single variable. There is no loss in generality in assuming that $c > 0$. We assume the boundary conditions

$$f(-\infty) = f(\infty) = 0 \quad , \quad g(\infty) = 1.$$

The travelling wave solution of (2.1) consists of functions f and g of $z = x - ct$ such that

$$\begin{cases} cg' - fg + g'' = 0, \\ cf' + fg - rf = 0, \end{cases} \tag{3.3}$$

For $c > 0$, by change of variable $g' = l$ we write (3.3) as

$$\begin{cases} g' = l \\ l' = -cl + fg, \\ f' = \frac{f}{c}[-g + r], \end{cases} \tag{3.4}$$

3.1.1 Equilibria

The equilibrium points of the reduced system satisfies $g' = l' = f' = 0$. The following theorem shows the obtained results.

Theorem 3.2. *The system* (3.3) *admits two positive equilibrium* $E_1 = (1, 0, 0)$ *and* $E_2 = (r, 0, 0)$.

Proof. We put $E^* = (g^*, l^*, f^*)$ arbitrary equilibrium

$$\begin{cases} 0 = l^* \\ \\ 0 = -cl^* + f^*g^*, \\ \\ 0 = \frac{f^*}{c}[-g^* + r], \end{cases}$$

Clarly,by the first equation we get $l^* = 0$, now by the third equation we find $f^* = 0$ or $g^* = r$.

1. If $f^* = 0$, we get $g^* = 1$, hence $E_1 = (1, 0, 0)$ is the first equilibrium .

2. If $g^* = r$, we find $f^* = 0$ the equilibrium given by $E_2 = (r, 0, 0)$, with $r < 1$.

□

3.1.2 Stability

Let $E = (g, l, f)$ be an arbitrary equilibrium, then, the Jacobian matrix of the system (3.4) at E is given by

$$J_{(g,l,f)} = \begin{pmatrix} 0 & 1 & 0 \\ f & -c & g \\ -\frac{f}{c} & 0 & \frac{1}{c}[r - g] \end{pmatrix}.$$

At first, we show the stability of E_1.

Theorem 3.3. *The equilibrium E_1 is unstable if $r > 1$.*

Proof. By evaluating this Jacobian matrix at E_1, we obtain

$$J_{(1,0,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -c & 1 \\ 0 & 0 & \frac{1}{c}[r - 1] \end{pmatrix}.$$

The corresponding characteristic polynomial is given by

$$p(\lambda) = -\lambda(-c - \lambda)(-\lambda + \frac{1}{c}[r - 1]) = 0.$$

If $r > 1$, we get $\lambda_1 = -c < 0$ and $\lambda_2 = \frac{1}{c}[r - 1] > 0$. Hence $E_1 = (1, 0, 0)$ is unstable. Morever, we have the following results Morever,if $r < 1$, we get $\lambda_1 = -c < 0$ and $\lambda_2 = \frac{1}{c}[r - 1] < 0$ but $\lambda_3 = 0$, so in this case, there is nothing that can be said about the stability of $E_1 = (1, 0, 0)$. □

Now we move to stability of $E_2 = (r, 0, 0)$, we have the following theorem :

Theorem 3.4. *Tthe local stability analysis do not give any information on the stability of E_2.*

Proof. By evaluating this Jacobian matrix at E_2, we obtain

$$J_{(1,0,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -c & r \\ 0 & 0 & 0 \end{pmatrix}.$$

clarly $\lambda_2 = -c < 0$ is negative eigenvalue, but $\lambda_{1,3} = 0$ so in this case, there is nothing that can be said about the stability of $E_2 = (r, 0, 0)$. □

Therefore, we have the following results,

Remarks 3.5. *If $r < 1$ there we cannot deduce the stability of the equilibrium and there fore we cannot deduce the existence of a traveling waves solutions. From the best of our knowlodge, we conldn't determin a Lyapunove function to prove the stability. Also, we notice that the change of variable performed in (2.4),(2.5) and (2.6) play an important role in determinig the existence of the travlling wave solution. This change of variables cann't be preformed in the case of $d_1 \neq 0$ and $d_2 = 0$.*

□

3.2 Numerical Simulation

In this section, we carry out the numerical simulations of the proposed model to proved existence of traveling waves solutions. For the initial conditions $S(0,x) = 4$ and $I(0,x) = 2$,

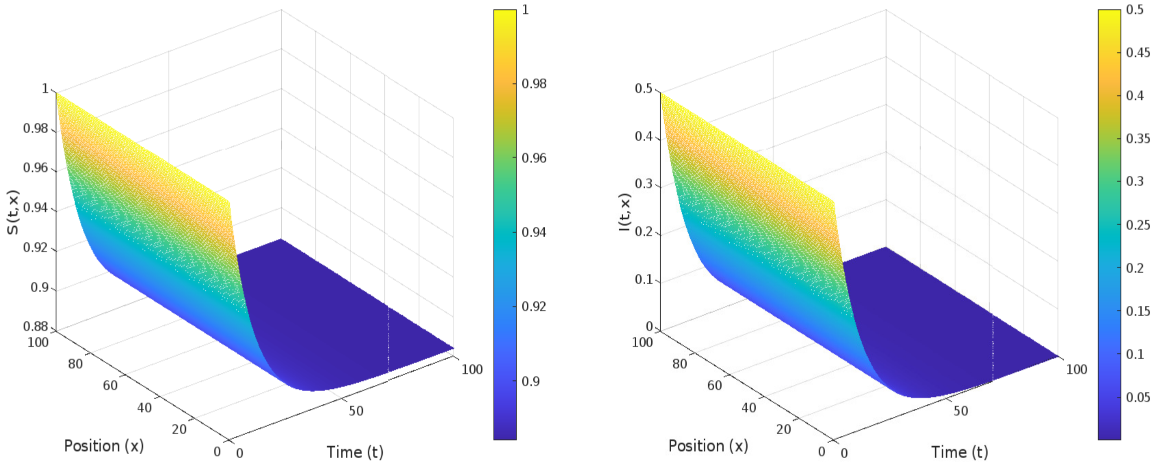

Fig. 8. Numerical simulations of solutions for system (3.1),where $r = 5 > 1$ and $R_0 = 0.2 < 1$,if $\gamma = 0.1$ and $\beta = 0.02$.

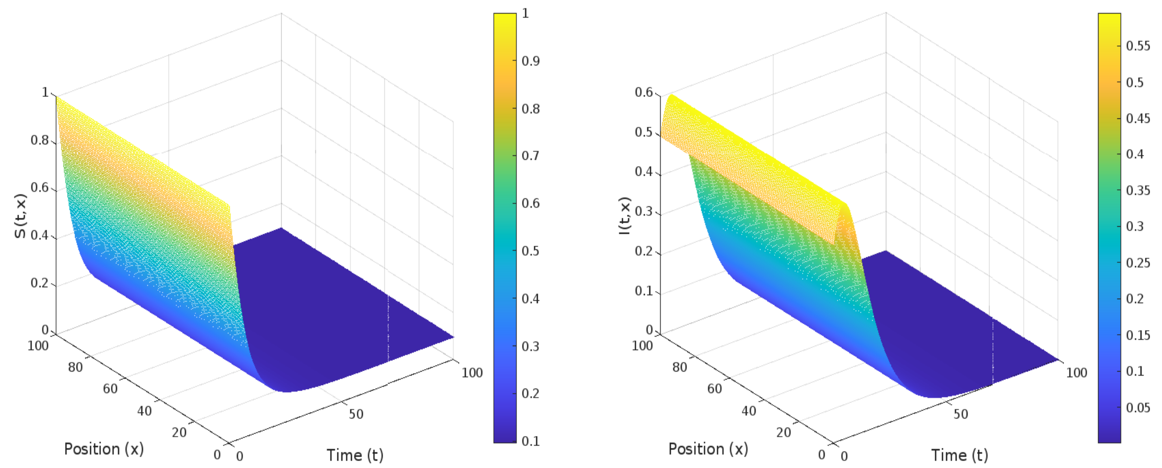

Fig. 9. Numerical simulations of solutions for system (3.1), where $r = 0.6 < 1$ and $R_0 = 1.66 > 1$, if $\gamma = 0.12$ and $\beta = 0.2$.

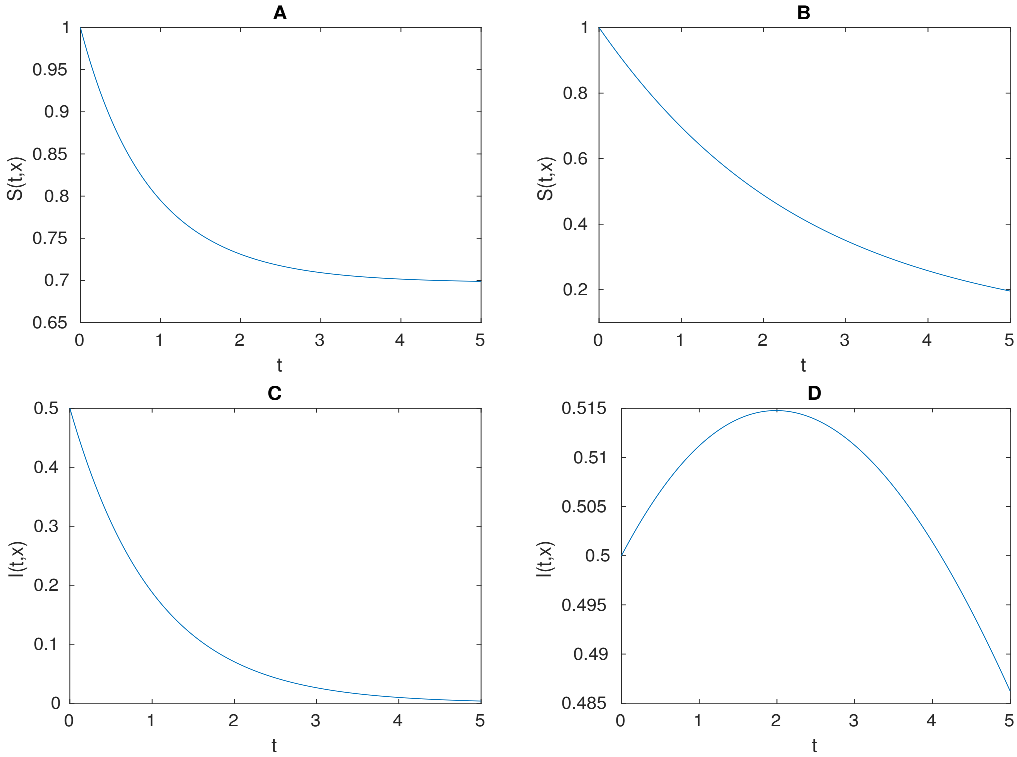

Fig. 10. The global dynamics of the system (3.1) compared to time (t), where for the left hand figure, we consider $r > 1$ if $\gamma = 0.3$, and for the right hand figure, we let $r<1$ if $\gamma = 0.1$.

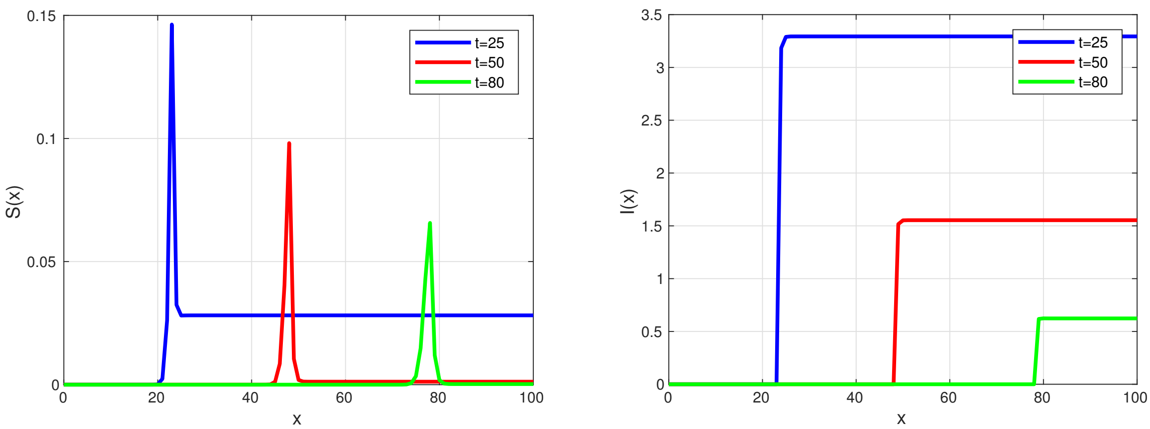

Fig. 11. The existence of traveling wave observed in system (3.1) with parameters: β=0.5 , γ = 0.3 and $d_2 = 0.1$, where $r = 0.6 < 1$.

Remarks 3.6. *If $r < 1$, then there is traveling waves solutions for the system (3.1), which determined numericlly.*

Data availability No data sets are associated with this manuscript.

Conflict of interest statement: The author declare no conflict of interest.

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