DOI: 10.22481/10.22481/intermaths.v2i2.9481
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Article | Artigo

INTERMATHS
ISSN 2675-8318
Vol. 2, N. 2, Jul - Dez 2021, p. 23 – 38

Cubic Hermite finite element method for nonlinear Black-Scholes equation governing European options

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Método dos elementos finitos baseado em polinómios de Hermite cúbicos, para resolução da equação de Black-Scholes não linear com opções europeias

Resumo
Foi desenvolvido um algoritmo numérico para resolver uma equação diferencial parcial generalizada de Black-Scholes, que surge na precificação de opções europeias, considerando os custos de transação. O método Crank-Nicolson é usado para discretizar no tempo e o método de interpolação cúbica de Hermite para discretizar no espaço. A eficiência e precisão do método proposto são testadas numericamente e, os resultados confirmam o comportamento teórico das soluções, que também se encontra em boa concordância com a solução exata.

Palavras-chave: Black-Scholes não linear, Método dos elementos finitos, Crank-Nicolson, Polinómios de Hermite.

Abstract
A numerical algorithm for solving a generalized Black-Scholes partial differential equation, which arises in European option pricing considering transaction costs is developed. The Crank-Nicolson method is used to discretize in the temporal direction and the Hermite cubic interpolation method to discretize in the spatial direction. The efficiency and accuracy of the proposed method are tested numerically, and the results confirm the theoretical behaviour of the solutions, which is also found to be in good agreement with the exact solution.

Keywords: Nonlinear Black-Scholes, Finite Element Method, Crank-Nicolson, Hermite Polynomials.

MSC: 65M60, 34K28, 35K55.

Submetido em: 07 de setembro de 2021 – **Aceito em**: 09 de dezembro de 2021

1 INTRODUCTION

The valuation of options based on stochastic processes dates back to 1877, when Charles Castelli wrote the book entitled "The Theory of Option in Stocks and Shares". Two decades later, Louis Bachelier, in his dissertation "Théorie de la spéculation", presented the first analytical way of calculating the price of an option. Subsequently, in 1955, in an unpublished manuscript entitled "Brownian Motion in the Stock Market", a professor at the Massachusetts Institute of Technology (MIT), Paul Samuelson, 1970 Nobel Prize in Economics, showed that the asset price can be modeled by a stochastic process called the Brownian Geometric Motion. In 1962, A. James Boness presented a dissertation entitled "Theory and Measurement of Stock Option Value", where he announced an option evaluation model that represented a great step forward from his predecessors and served as the basis for the work later developed by Black and Scholes. The Black-Scholes model [9] is a well-known popular model which is used to calculate the price of European options. Since its inception in 1973 by Fischer Black and Myron Scholes, it remains one of the preferred models and provides a basis for the theory of financial options. The linear Black-Scholes equation is given by

$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \qquad S > 0, \qquad t \in]0, T[, \tag{1}$$

where V is the option value, T the expiry time, S the underlying asset price, σ the volatility and r the riskless interest rate.
Equation (1) permits the evaluation of the price of a European option under the assumptions listed below:

1. the value of the financial asset underlying the option can be modeled by a geometric Brownian motion;
2. there are no transaction costs associated with the management of financial asset portfolios, nor fees payable in the market;
3. the market does not allow arbitrage opportunities;
4. short selling is permitted;
5. there is a risk-free rate that is constant throughout the life of the option and it is possible to lend and borrow at that same rate any financial asset;
6. the volatility of the underlying asset is known and remains constant throughout the option lifespan;
7. the transaction of the financial asset is made on a continuous basis and changes in its price are also on a continuous basis;

8. fractional parts of an asset can be obtained;

9. the financial asset does not pay dividends during the option lifespan.

The classical Black-Scholes model is notable for its explicit closed form solution of European style options (call and put options). Many researchers have attempted to obtain the solution of the Black-Scholes Equation analytically and/or numerically, thereby adopting and using various methods. For a survey of the classical methods (Binomial, Monte Carlo, Finite Differences), we refer the reader to the survey book [10].

Bohner, Sánchez and Rodríguez [8] applied the Adomian Decomposition Method. Khatskevich [7] obtained the option execution price in the form of a Legendre polynomial series. In [6] Edeki, Ugbebor and Owoloko proposed a method referred to as the Projected Differential Transformation Method. Altough Finite Element Methods may seem at first glance unnecessarily complex for finance, where a large class of problems are one dimensional in space, they are very flexible and give good approximations (see, for example [5]).

The Black-Scholes equation is very effective in a market without transaction costs, but transaction costs may arise when trading securities. Although they are small in general, they can lead to an increase in the option price in which case the Black-Scholes pricing methodology will no longer be valid since perfect hedging is impossible.Consequentely, different models have been proposed to modify equation (1) in order to accommodate transaction costs, such as those in [11, 12, 13]. In these models, the constant volatility is replaced by a modified volatility which can depend on time, on the asset price, on the option value and its derivatives. The resulting model is a nonlinear equation in nondivergence form. For the general nonlinear Black-Scholes equation an explicit solution is unknown and the numerical tecniques available are far less than for the linear model.

In 2003, During, Fournié and Jungel ([4]) deduced a high order compact finite difference scheme for the nonlinear Black-Scholes model with transaction costs presented in [12] which proved to be unconditionally stable and non-oscillatory.

J. Ankudinova and M.Ehrhardt [14] made a comparative study between models with transaction costs and the linear model. The influence of transaction costs modeled by the volatilities given by the Leland, Barles and Soner and Krakta models was calculated by the Crank-Nicolson method in time and by the finite difference method in space. They studied the difference between the price of the European call option with transaction costs and the European call option price without transaction costs. Their numerical results indicate an economically significant price deviation between the standard (linear) Black-Scholes model and nonlinear models.

Company, Jódar and Pintos constructed and analysed another finite difference scheme for the nonlinear Black-Scholes model deduced by Barnes and Soner. Consistency and stability were studied and some illustrative examples were presented.

D. Lesmana and S. Wang [3] developed a numerical method for a nonlinear parabolic partial differential equation resulting from the pricing of European options under transaction costs. Their method is based on an upwind finite difference scheme for spatial discretization and on a totally implicit scheme for time discretization. The convergence of the solution of the discretized system to the viscosity solution of the continuous problem was proved. They proposed a Newton's iterative method to solve the resulting nonlinear algebraic system and showed that the Jacobian matrix of the nonlinear system is an $M-$ matrix and thus the solution of the system linearized by an iterative scheme is numerically stable. Simulations were performed to illustrate the accuracy and usefulness of the method, and it was seen that the convergence orders are about 1.6 and 2 in the discrete norms L_∞ and L_2, respectively. The results also showed that the price of a European option is an increasing function of the parameter a of the transaction cost.

Almeida et al., in ([15],[16],[17]), established convergence, properties and error bounds for the fully discrete solutions of a class of nonlinear equations of reaction-diffusion nonlocal type, using a linearised Crank-Nicolson-Galerkin finite element method with polynomial approximations of arbitrary degree.

In [2], Böhmer presented a quite simple and intuitive nonstandard C^1 finite element method to approximate the classical solution of a general fully nonlinear second order elliptic equation. Using some intricate consistency and stability arguments he proved, under certain conditions, the existence of a unique solution also derived Optimal order error estimates when u is sufficiently smooth.

In this work, we will apply Böhmer's method with a C^1 cubic Hermite basis for the space discretization and the Crank-Nicolson method to discretize in the temporal direction. Some examples will be presented to test the efficiency and accuracy of the proposed method. The remainder of this paper is organized as follows. In Section 2, the problem is described, considering some transaction cost models for European options. In Section 3, we define the problem as a general nonlinear partial differential equation in nondivergent form. In Section 4, we construct the discretization in spatial direction with the Hermite interpolation method in a uniform mesh and, in Section 5, we discretize in temporal direction with the Crank-Nicolson method. In Section 6, we obtain and compare the approximate numerical solutions. Finally, in Section 7, we draw some conclusions.

2 NONLINEAR BLACK-SCHOLES MODEL

As has been pointed out by several authors [12, 14, 10] the Black-Scholes model requires a portfolio adjustment in order to protect a risk-free hedge. In the presence of transaction cost, this adjustment is likely to be more expensive, since an infinite number of transactions is required [19]. But the hedger needs to find the balance between the transaction costs that are needed to rebalance the portfolio and the implicit hedging error costs. As a result of

this "imperfect" coverage, the option can be overly underestimated, where the risk-free profit obtained by the arbitrator is offset by the transaction cost so that there is no single equilibrium price but a viable price range. It has been demonstrated that in a transaction market there is no replicator portfolio for a European type call option and the portfolio is required to dominate rather than replicate the option value (see [12]).
Soner, Shreve and Cvitanič [18] have proved that the minimum coverage portfolio of a financial option is trivial, so efforts have been made to ease the condition coverage criterion to better replicate pay-off of derivative securities. Because of the presence of transaction costs (see [12], [13], [11]) the classical model results in a strongly or wholly nonlinear and possibly degenerate parabolic type diffusion equation where the volatility σ may depend on the time t, the price S or on other derivatives of the option price V. In this work, we study the nonlinear Black-Scholes equation with some transaction cost models for European options, with σ a non-constant modified volatility function

$$\tilde{\sigma}^2 := \tilde{\sigma}^2(t, S, V_S, V_{SS}).$$

In this way, Equation (1) becomes the following nonlinear Black-Scholes equation

$$0 = V_t + \frac{1}{2}\tilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} + rSV_S - rV, \qquad S > 0, \qquad t \in (0, T). \tag{2}$$

A European call option allows the buyer to buy an asset of value S for a value K on the maturity date T, while a European put option allows the holder to sell an asset of value S for a value K on the maturity date T. Since the option can only be exercised on maturity, we complement Equation (2) with the following conditions, in order to avoid arbitrage:

European call option:

$$V(S, T) = \max\{S - K, 0\}, \qquad \text{when} \quad S \geq 0 \tag{3}$$

$$\lim_{S \to \infty} \frac{V(S, t)}{S - Ke^{-r(T-t)}} = 1, \qquad \text{for} \quad t \in [0, T] \tag{4}$$

$$V(0, t) = 0, \qquad \text{for} \quad t \in [0, T] \tag{5}$$

$$\lim_{S \to \infty} V_S(S, t) = 1, \qquad \text{for} \quad t \in [0, T] \tag{6}$$

European put option:

$$V(S, T) = \max\{K - S, 0\}, \qquad \text{when} \quad S \geq 0 \tag{7}$$

$$V(0,t) = Ke^{-r(T-t)}, \qquad \text{for} \ \ t \in [0,T] \tag{8}$$

$$\lim_{S\to\infty} V(S,t) = 0, \qquad \text{for} \ \ t \in [0,T] \tag{9}$$

$$\lim_{S\to\infty} V_S(S,t) = 0, \qquad \text{for} \ \ t \in [0,T] \tag{10}$$

2.1 Leland's model

In [11], Leland deduces that the option price is the solution of the nonlinear Black-Scholes Equation (2), with modified volatility given by

$$\tilde{\sigma}^2 = \sigma^2\Big(1 + Le \times sign(V_{SS})\Big). \tag{11}$$

In (11) Le is Leland's number, which is given by

$$Le = \sqrt{\frac{2}{\pi}}\left(\frac{k}{\sigma\sqrt{\delta t}}\right) \tag{12}$$

where δt is the interval between two successive revisions of the portfolio, k is the round trip transaction cost per transacted monetary unit and σ represents the historical volatility.
By a different process, Boyle and Vorst [1] deduced a similar modified volatility, given by

$$\tilde{\sigma}^2 = \sigma^2\left(1 + Le\sqrt{\frac{\pi}{2}} sign(V_{SS})\right). \tag{13}$$

However, δt (12) represents the interval between two successive portfolio reconstructions and not the transaction frequency as in (13) (see [12]).

2.2 Barles' and Soner's model

Barles and Soner developed a complex model based on the Hedges and Neuberger [20] utility function approach. In order to simplify the calculations, they proposed in [12], the volatility model

$$\tilde{\sigma}^2 = \sigma^2\left(1 + e^{r(T-t)}a^2S^2V_{SS}\right), \tag{14}$$

where σ is the historical volatility and $a = \frac{k}{\sqrt{\varepsilon}}$. They proved the existence of a viscosity solution for the European option with the volatility given by (14) and their numerical results indicate an economically significant price difference between the standard Black-Scholes model and the nonlinear model with transaction costs.

2.3 Kratka's model

The model proposed by Krakta in [13] minimizes the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. In this way, the portfolio is well protected with the Risk Adjusted Pricing Methodoly (RAPM) and the modified volatility is given by

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + 3\left(\frac{C^2 M}{2\pi} S V_{SS}\right)^{\frac{1}{3}}\right), \tag{15}$$

where $M \geq 0$ is the measure of the transaction cost and $C \geq 0$ the risk premium.

It should be noted that the nonlinear transaction cost models described above are all consistent with the linear model if the additional parameters for the transaction cost are zero.

3 NONLINEAR GENERAL EQUATION

In this work, we study equations of the form:

$$u_t = c_0 u_{xx} + c_1 u_x + c_2 u + f, \qquad \text{with} \qquad a < x < b, \qquad 0 < t < T, \tag{16}$$

under the initial and boundary conditions

$$u(x, 0) = u_0(x), \qquad a < x < b, \tag{17}$$

$$\begin{cases} u(a,t) = g_1(t) \\ u(b,t) = g_2(t) \end{cases} \quad \text{and} \quad \begin{cases} u_x(a,t) = g_3(t) \\ u_x(b,t) = g_4(t) \end{cases} \quad 0 < t < T, \tag{18}$$

where $c_0 = c_0(x, t, u, u_x, u_{xx})$, $c_1 = c_1(x, t, u, u_x)$, $c_2 = c_2(x, t, u)$, $f = f(x, t)$, $g_1(t)$, $g_2(t)$, $g_3(t)$, $g_4(t)$ and $u_0(x)$ are known real bounded functions.

Note that (16)-(18) is a general model which includes the problem under study. The transformation $u(x, t) = V(S, T - t)$ transforms (2) into (16) with $c_0 = \frac{1}{2}\tilde{\sigma}^2 x^2$, $c_1 = rx$, $c_2 = -r$ and $f = 0$, and the initial condition becomes

$$u(x, 0) = \max\{K - x, 0\} \quad \text{or} \quad u(x, 0) = \max\{x - K, 0\}$$

by (3) and (7).

For a call option, condition (5) is satisfied considering $g_1(t) = 0$. For b sufficient large conditions, (4) and (6) can be approximated by $g_2(t) = b - Ke^{-rt}$ and $g_4(t) = 1$.

Since we require another condition, motivated by the behavior of the solution for the linear equation, we consider $g_3(t) = 0$.

For a put option, (8), (9) and (10) imply that $g_1(t) = Ke^{-rt}$, $g_2(t) = 0$, $g_3(t) = -1$ and

$g_4(t) = 0.$

Let w be a test function. Multiplying (16) by w and integrating in]a,b[, we obtain

$$\int_a^b u_t w\, dx - \int_a^b c_0 u_{xx} w\, dx - \int_a^b c_1 u_x w\, dx - \int_a^b c_2 u w\, dx = \int_a^b f w\, dx. \tag{19}$$

Since c_0 depends on u_{xx}, integration by parts is useless. For relation (19) to make sense, we most have u, u_t, u_x and $u_{xx} \in L_2(a,b)$, that is, u must be in $C^1(a,b)$, for $t \in]0,T]$. According to the conditions in (18), we choose the test function space to be

$$V_0 = \{w, w_x, w_{xx} \in L_2(a,b) : w(a) = w(b) = w_x(a) = w_x(b) = 0\},$$

and for the space solution we consider

$$V = \{u, u_t, u_x, u_{xx} \in L_2(a,b) : u(a,t) = g_1(t), u(b,t) = g_2(t), u_x(a,t) = g_3(t), u_x(b,t) = g_4(t)\,,\ \text{for all } t \in [0,T]\},$$

4 DISCRETIZATION IN SPACE

We consider the discretization $a = x_0 < x_1 < \cdots < x_{m+1} = b$ of $[a,b]$ with spacing h and since continuity in C^1 is required, we define for each node x_i, two Hermite interpolation polynomials, $\varphi_i(x)$ and $\psi_i(x)$. The Hermite interpolation polynomials have support $[x_{i-1}, x_{i+1}]$ and are defined by

$$\varphi_i(x) = \begin{cases} -2\left(\frac{x-x_i}{h}\right)^3 - 3\left(\frac{x-x_i}{h}\right)^2 + 1, & x \in [x_{i-1}, x_i[, \\ 2\left(\frac{x-x_i}{h}\right)^3 + 3\left(\frac{x-x_i}{h}\right)^2 + 1, & x \in [x_i, x_{i+1}], \end{cases} \tag{20}$$

and

$$\psi_i(x) = \begin{cases} \frac{(x-x_i)^3}{h^2} + \frac{(x-x_i)^2}{h} + (x - x_i)\,, & x \in [x_{i-1}, x_i[\\ \frac{(x-x_i)^3}{h^2} - \frac{(x-x_i)^2}{h} - (x - x_i)\,, & x \in [x_i, x_{i+1}] \end{cases} \tag{21}$$

The Hermite cubic polynomials satisfy the following interpolation properties:

$$\varphi_j(x_i) = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}, \qquad \varphi_j'(x_i) = 0, \tag{22}$$

$$\psi_j(x_i) = 0, \quad \psi_j'(x_i) = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}, i, j = 0, \ldots, m+1 \tag{23}$$

and hence they satisfy the required continuity conditions.

Let $\mathcal{H} = \langle \varphi_0, \psi_0, \varphi_1, \psi_1, \cdots, \varphi_{m+1}, \psi_{m+1} \rangle$ be the vector subspace generated by the $2m+2$

elements of the Hermite basis. Let us consider the test function in

$$V_{0m} = \{w_m(x) \in \mathcal{H} : w_m(a) = w_m(b) = (w_m)_x(a) = (w_m)_x(b) = 0\} \subset V_0$$

and the approximate solution in

$$\begin{aligned} V_m &= \{u_m(x,t) \in \mathcal{H} : u_m(a,t) = g_1(t),\ u_m(b,t) = g_2(t), \\ (u_m)_x(a,t) &= g_3(t),\ (u_m)_x(b,t) = g_4(t), \quad \text{for all } t \in [0,t]\} \subset V. \end{aligned}$$

A function $u_m \in V_m$ is said to be an approximate solution of (19) if, for each $t \in]0,T]$, it satisfies

$$\begin{aligned} &\int_a^b (u_m)_t w_m\,dx - \int_a^b c_0(u_m)_{xx} w_m\,dx - \int_a^b c_1(u_m)_x w_m\,dx \\ &- \int_a^b c_2(u_m) w_m\,dx = \int_a^b f w_m\,dx, \qquad \text{for all } w_m \in V_{0m}. \end{aligned} \tag{24}$$

Any function $w_m \in V_{0m}$ can be written as

$$w_m(x) = \sum_{i=1}^{m} W_i \varphi_i(x) + Z_i \psi_i(x), \tag{25}$$

and any function $u_m \in V_m$ can be written as

$$\begin{aligned} u_m(x,t) \quad &= \varphi_0(x) g_1(t) + \psi_0(x) g_3(t) + \sum_{i=1}^{m} \varphi_i(x) U_i(t) + \psi_i(x) V_i(t) \\ &+ \varphi_{m+1}(x) g_2(t) + \psi_{m+1}(x) g_4(t). \end{aligned} \tag{26}$$

Substituting (25) and (26) in equation (24) and simplifying the expressions, we obtain a system of ordinary differential equations, which can be written in matrix form:

$$MU(t)' - A(U(t))U(t) - B(U(t))U(t) - C(U(t))U(t) = F(t) + D(t), \tag{27}$$

with the unknown

$$U(t) = [U_1, ..., U_m, V_1, ..., V_m]^T.$$

and where matrices are given by:

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix},$$

$$M_1(i,j)=\int_a^b \varphi_i\,\varphi_j\;dx,\quad M_2(i,j)=\int_a^b \varphi_i\,\psi_j\;dx,\quad M_3(i,j)=\int_a^b \psi_i\,\psi_j\;dx$$

$$A=\left[\begin{array}{cc} A_1 & A_2\\ A_3 & A_4\end{array}\right],$$

$$A_1(i,j)=\int_a^b c_0(U)\varphi_i\,\varphi_j''\;dx,\quad A_2(i,j)=\int_a^b c_0(U)\varphi_i\,\psi_j''\;dx,$$

$$A_3(i,j)=\int_a^b c_0(U)\psi_i\,\varphi_j''\;dx,\quad A_4(i,j)=\int_a^b c_0(U)\psi_i\,\psi_j''\;dx$$

$$B=\left[\begin{array}{cc} B_1 & B_2\\ B_3 & B_4\end{array}\right],$$

$$B_1(i,j)=\int_a^b c_1(U)\varphi_i\,\varphi_j'\;dx,\quad B_2(i,j)=\int_a^b c_1(U)\varphi_i\,\psi_j'\;dx,\varphi_j'\;dx$$

$$B_3(i,j)=\int_a^b c_1(U)\psi_i\,\varphi_j'\;dx,\quad B_4(i,j)=\int_a^b c_1(U)\psi_i\,\psi_j'\;dx$$

$$C=\left[\begin{array}{cc} C_1 & C_2\\ C_2^T & C_3\end{array}\right],$$

$$C_1(i,j)=\int_a^b c_2(U)\varphi_i\,\varphi_j\;dx,\quad C_2(i,j)=\int_a^b c_2(U)\varphi_i\,\psi_j\;dx,$$

$$C_3(i,j)=\int_a^b c_2(U)\psi_i\,\psi_j\;dx$$

$$F=\left[\begin{array}{c} F_1\\ F_2\end{array}\right],\quad F_1(i)=\int_a^b f\varphi_i\,dx,\quad F_2(i)=\int_a^b f\psi_i\;dx,$$

$$D=\left[\begin{array}{c} D_1\\ D_2\end{array}\right]$$

$$
\begin{aligned}
&D_1(i) = \\
&-M_1(i,0)g_1'(t) - M_1(i,n+1)g_2'(t) - M_2(i,0)g_3'(t) - M_2(i,n+1)g_4'(t) \\
&+A_1(i,0)g_1(t) + A_1(i,n+1)g_2(t) + A_2(i,0)g_3(t) + A_2(i,n+1)g_4(t) \\
&+B_1(i,0)g_1(t) + B_1(i,n+1)g_2(t) + B_2(i,0)g_3(t) + B_2(i,n+1)g_4(t) \\
&+C_1(i,0)g_1(t) + C_1(i,n+1)g_2(t) + C_2(i,0)g_3(t) + C_2(i,n+1)g_4(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_2(i) = \\
&-M_2^T(i,0)g_1'(t) - M_2^T(i,n+1)g_2'(t) - M_3(i,0)g_3'(t) - M_3(i,n+1)g_4'(t) \\
&+A_3(i,0)g_1(t) + A_3(i,n+1)g_2(t) + A_4(i,0)g_3(t) + A_4(i,n+1)g_4(t) \\
&+B_3(i,0)g_1(t) + B_3(i,n+1)g_2(t) + B_4(i,0)g_3(t) + B_4(i,n+1)g_4(t) \\
&+C_2^T(i,0)g_1(t) + C_2^T(i,n+1)g_2(t) + C_3(i,0)g_3(t) + C_3(i,n+1)g_4(t)
\end{aligned}
$$

In general, the solution $U(t)$ is not explicitly known for all $t \geq 0$, so it is necessary to use a numerical method to obtain an approximate solution.

5 DISCRETIZATION IN TIME

Let us now consider the partition $0 = t_0 < t_1 < \cdots < t_N = T$, with step δ, of $[0, T]$. By the Cranck-Nicolson method, evaluating (27) in $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$ and using the approximations

$$U'(t_{n+\frac{1}{2}}) \approx \frac{U(t_{n+1}) - U(t_n)}{\delta} = \frac{U_{n+1} - U_n}{\delta} \tag{28}$$

and

$$U(t_{n+\frac{1}{2}}) \approx \frac{U(t_{n+1}) + U(t_n)}{2} = \frac{U_{n+1} + U_n}{2}, \tag{29}$$

we obtain

$$
\begin{aligned}
(2M + \delta\,(A_{n+1} + B_{n+1} + C_{n+1})U_{n+1} = (2M - \delta\,(A_{n+1} + B_{n+1} + C_{n+1}))U_n \\
+2\delta F_{n+1/2} - 2\delta D_{n+1/2}, \quad n = 0, 1, \cdots, N-1.
\end{aligned} \tag{30}
$$

Since the function $f(x,t)$ is known, $F_{n+1/2} = F(x, t_{n+1/2})$ is also known for all n. Then, for each $n = 0, 1, \cdots, N-1$, the system of algebraic equations (30), may be solved using the fixed point scheme:

$$\begin{aligned}&\left[2M+\delta\left(A_{n+1}^{(k)}+B_{n+1}^{(k)}+C_{n+1}^{(k)}\right)\right]U_{n+1}^{(k+1)}=(2MU_n\\&-\delta\left(A_{n+1}^{(k)}+B_{n+1}^{(k)}+C_{n+1}^{(k)}\right))U_n+2\delta F_{n+1/2}-2\delta D_{n+1/2}\end{aligned}\tag{31}$$

$$U_{n+1}^{(0)}=U_n \quad and \quad n=1,2,\ldots,N \quad k=1,2,\ldots.$$

6 NUMERICAL RESULTS

In this section, we present the results of a Matlab implementation of the theory. First we validate the code by simulating the linear equation and calculating the error and then we compare the solutions of the nonlinear equation obtained with the different modified volatilities presented.

Example 1: In order to calculate the value of the call and put options, we transform the Black-Scholes equation in(1) into an equivalent equation which is easier to solve, considering the new variables $y=\log(S)$ and $\tau=T-t$, and apply the separation of variables method to the new equation to obtain the solution,

$$\begin{aligned}V(S,t) \quad = \quad & Ae^{rt+\sqrt{\lambda}\ln(S)+\sqrt{\lambda}(r-\frac{1}{2}\sigma^2)(T-t)+\frac{\lambda\sigma^2}{2}(T-t)}\\&+Be^{-rt-\sqrt{\lambda}\ln(S)-\sqrt{\lambda}(r-\frac{1}{2}\sigma^2)(T-t)+\frac{\lambda\sigma^2}{2}(T-t)},\end{aligned}\tag{32}$$

where

$$A,B\in\mathbb{R} \quad \text{and} \quad \lambda>0.$$

Condition (5) implies that $B=0$, but conditions (3) and (6) are not satisfied.
We simulated equation (1) with $r=0.1$, $\sigma=0.2$, $\lambda=25$, $A=1$, $a=0$, $b=1$, $T=1$, $g_1(t)=0$, $g_2(t)=e^{(0.5-0.4t)}$, $g_3(t)=0$, $g_4(t)=5e^{(0.5-0.4t)}$ and $u_0(x)=e^{5ln(x)+0.5}$.
In figure 1,the picture on the left shows the convergence error for h, with $\delta=0.0001$ and varying values of $h=0.333,0.170,0.1,0.056,0.032,0.018$. In the picture on the right we have the convergence error for δ, considering $h=0.001$ and varying values of $\delta=0.1,0.01,0.001$.

We observe that convergence orders are optimal, that is $\mathcal{O}(h^4)$ for the solution, $\mathcal{O}(h^3)$ for its derivative and $\mathcal{O}(\delta^2)$ for both solution and derivative.

Example 2: The analytical solution of the linear Black-Scholes equation in (1), where both r and σ are constant and satisfy condition (3), is given by the well-known formula [10]

$$V(S,t)=SN(d_1)-Ke^{-r(T-t)}N(d_2)\tag{33}$$

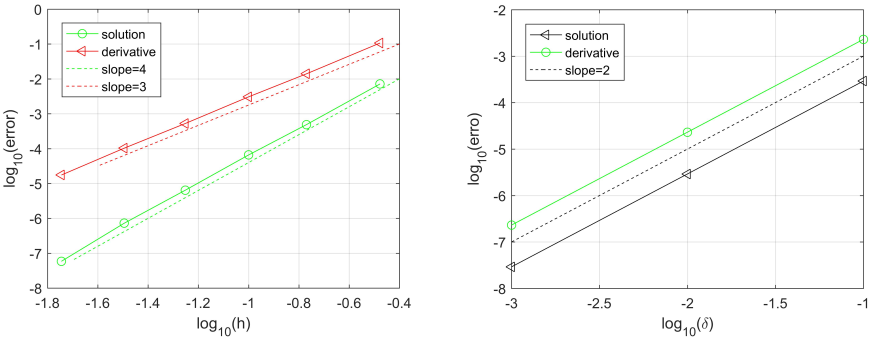

Figure 1: Convergence analysis for h and for δ in example 1.

where $N(x) = (1/2\pi) \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy$, $x \in \mathbb{R}$ is the cumulative distribution function of $N(0,1)$, $d_1 = \{\log(S/K) + (r + \sigma^2/2)(t-T)\}/\left(\sigma\sqrt{t-T}\right)$ and $d_2 = d_1 - \sigma\sqrt{(t-T)}$. Knowing the exact explicit solution permitted us to calculate the exact error of the approximations. We simulated equation (16) using the parameters $T = 1$, $r = 0.1$, $\sigma = 0.2$, $a = 0.4$, $b = 1$, $K = 0.4$, $T = 1$, $g_1(t) = 0$, $g_2(t) = 1 - 0.4e^{rt}$, $g_3(t) = 0$, $g_4(t) = 1$ and $u_0(x) = \max\{x - 0.4, 0\}$.
In Figure 2, we present the solution obtained with $h = 0.01$ and $\delta = 0.001$, at some instants. We may observe that the behaviour is similar to the behaviour of the exact solution.
In figure 3, the convergence error for h is studied, where we consider a fixed value $\delta = 0.001$ and different values of $h = 0.1, 0.01, 0.001$. For each value of h, we calculated the error in the $L_2(a,b)$ norm and we collected the results in the graph presented.
From the graph we may conclude that the convergence is only of order 2. We suspect that

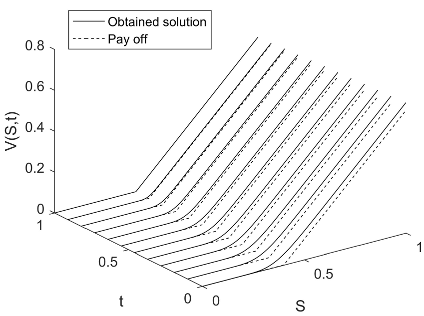

Figure 2: Obtained solution in example 2.

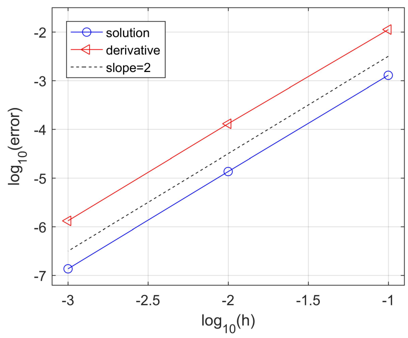

Figure 3: Convergence analysis for h in example 2.

this behaviour is due to the fact that the solution is not regular, since in Example 1 the order of convergence is 4.

Example 3: In order to compare the behaviour of the solution for the different models presented, we simulated Equation (16) for each model for a European call. In Figure 4, we represent the solution and the first derivative of the nonlinear Black-Scholes equation for the different transaction cost models at $t = 0$. The parameters used are: $r = 0.2$, $\sigma = 0.2$,

$Le = 0.6,\quad M = 30,\quad C = 0.01,\quad a^2 = 0.4,\quad K = 0.4,\quad T = 1,\quad h = 0.1,\quad \delta = 0.001,$ $g_1(t) = 0,\quad g_2(t) = 1 - 0.4e^{rt},\quad g_3(t) = 0,\quad g_4(t) = 1$ and $u_0(x) = \max\{x - 0.4, 0\}$.

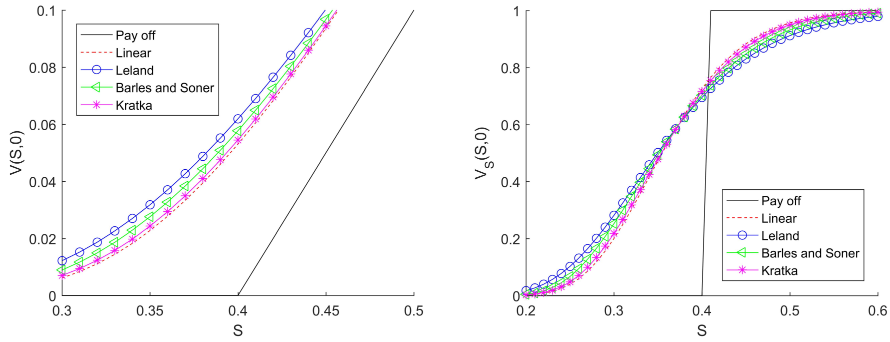

Figure 4: Solution (left) and its derivative (right) of the nonlinear Black-Scholes equation with different transaction costs for a European call option.

Example 4: Finally we simulated Equation (16) for each model for a European put. In each picture of Figure 5 we represented the solution and also the first derivative of the nonlinear Black-Scholes equation for the different transaction costs models. The parameters used are: $r = 0.2,\quad \sigma = 0.2,\quad Le = 0.6,\quad M = 30,\quad C = 0.01,\quad a^2 = 0.4,\quad K = 0.4,$ $T = 1,\quad h = 0.1,\quad \delta = 0.001,\quad g_1(t) = 0.4e^{rt} - 1,\quad g_2(t) =0,\quad g_3(t) = -1,\quad g_4(t) = 0$ and $u_0(x) = \max\{0.4 - x, 0\}$.

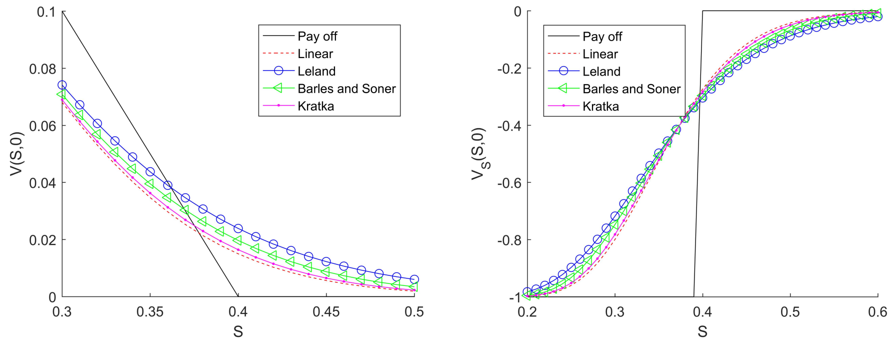

Figure 5: Solution (left) and its derivative (right) of the nonlinear Black-Scholes equation with different transaction costs for a European put option.

The chart shows that the difference between the various transaction cost models is not significant. At this point, with the given parameters, the Leland model provides the highest price, followed by the Barles and Soner model, the Kratka model, and finally the linear model with constant volatility without transaction costs. An analysis from the initial date to the maturity date permits us to conclude that the difference between the various models decreases as the expiration date approaches. This is an expected consequence of the decreasing necessity of portfolio adjustment and hence lower transaction costs closer the to expiry date. The difference is bigger at the beginning of the year, where the nonlinear price is higher than the linear price.

7 CONCLUSIONS

A finite element method based on Hermite polynomials to solve the nonlinear problem in the non-divergent form, in a domain with fixed boundaries, was presented. The program resulting from the implementation of this method in Matlab code was tested with the linear equation. The error and convergence were analysed in Examples 1 and 2. In Example 1, the finite element method has a convergence order of approximately 4 for the solution and 3 for its derivative, and the Crank-Nicolson method has order $\mathcal{O}(\delta^2)$, for the solution and its derivative. In Example 2, the Crank-Nicolson method presents a convergence order of 2, for both the solution and its derivative, while the finite element method has a convergence order of approximately 2 for the solution and 2 for its derivative, which does not fit the methods applied in this work. In Examples 3 and 4, the solution and the derivative of the nonlinear Black-Scholes equation were simulated with the different transaction cost models, taking into account the European-type call and put options. The study shows that the difference is not significant for all transaction cost models and it decreases the closer we are to the expiration date. As future work, we will carry out a similar study for American options.

ACKNOWLEDGEMENTS

This work was partially supported by the research projects: Grant NÂ UID/MAT/00212/2013 - financed by FEDER through the - Programa Operacional Factores de Competitividade, FCT - Fundação para a Ciência e a Tecnologia and Grant BID/ICI-FC/Santander Universidades-UBI/2016.

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