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Lyapunov instability for discontinuous differential equations

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Instabilidade de Lyapunov para equações diferenciais descontínuas

Resumo
O presente trabalho estuda a instabilidade de Lyapunov para equações diferenciais descontínuas através do uso da noção de solução de Carathéodory para equações diferenciais. A partir do primeiro teorema de instabilidade de Lyapunov e do teorema de instabilidade de Chetaev, que tratam da instabilidade para equações diferenciais ordinárias, dois resultados de instabilidade de Lyapunov para equações diferenciais descontínuas são obtidos.

Palavras-chave: Equações diferenciais descontínuas; Soluções de Carathéodory; Estabilidade de Lyapunov; Instabilidade.

Abstract
The present work studies the Lyapunov instability for discontinuous differential equations through the use of the notion of Carathéodory solution to differential equations. From Lyapunov's first instability theorem and Chetaev's instability theorem, which deal with instability to ordinary differential equations, two Lyapunov instability results for discontinuous differential equations are obtained.

Keywords: Discontinuous differential equations; Carathéodory solutions; Lyapunov stability; Instability.

MSC: 34A36; 34D20; 26A46.

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1 INTRODUCTION

Discontinuous differential equations are ordinary differential equations with the discontinuous right side and determine discontinuous systems. Such differential equations are treated, for example, in [1], [2], [3], [4], [5] and [6]. The study of the Lyapunov stability to discontinuous differential equations using the notion of Carathéodory solution can be found in [3] and [6].

Based on instability results for ordinary differential equations, the present work studies the instability for discontinuous systems determined by

$$\dot{x}(t) = f(t, x(t)) \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $f(t, 0) = 0$ for all $t \in [0, \infty)$. For this, the notion of Carathéodory solution to Eq. (1) is used here. Thus, from Lyapunov's first instability theorem (see [7, Theorem 9.16]) and Chetaev's instability theorem (see [7, Theorem 9.22]), two Lyapunov instability results are stablished to Eq. (1). Moreover, examples that illustrate the established results are considered.

The existence of Carathéodory solutions for (1) can be found in [8]. In [4] we can find the study on the continuation of solutions.

2 PRELIMINARIES

This section provides basic concepts and results that will be used in the development of the work.

2.1 Absolutely continuous functions

Absolutely continuous functions are treated, for example, in [9]. In the next definition, $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Definition 2.1. A function $x : [a, b] \to \mathbb{R}^n$ is called absolutely continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any countable collection of disjoint subintervals $[a_k, b_k]$ of $[a, b]$ satisfying

$$\sum (b_k - a_k) < \delta,$$

implies that

$$\sum \|x(b_k) - x(a_k)\| < \varepsilon.$$

We also define absolutely continuous functions on a given interval $I \subset \mathbb{R}$.

The work uses the notion of Lebesgue integral. Consider an interval $I \subset \mathbb{R}$. We say that a statement P holds almost everywhere (a.e.) on I, if the set N given by

$$N = \{t \in I : P \text{ does not hold at } t\}$$

has Lebesgue measure zero. A more complete approach to Lebesgue measure and integral can be found in [10]. As can be seen in [11], an absolutely continuous function $x : [a,b] \to \mathbb{R}^n$ is differentiable almost everywhere, and its derivative $\dot{x}(\cdot)$ is a Lebesgue integrable function. The Newton-Leibniz formula is also true; that is,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t)dt$$

for all $t_1, t_2 \in [a,b]$, $t_1 < t_2$.

2.2 Carathéodory solutions

Below we have the definition of Carathéodory solution to Eq. (1).

Definition 2.2. Consider an interval $I \subset \mathbb{R}$. It is said that a function $x : I \to \mathbb{R}^n$ is a Carathéodory solution to Eq. (1) on I if $x(t)$ is absolutely continuous and $\dot{x}(t) = f(t, x(t))$ for a.e. $t \in I$.

Suppose that $S(x_0)$ denotes the set of Carathéodory solutions $x(t)$ to Eq. (1) on $[0,\infty)$ with $x(0) = x_0$. If $f(t,0) = 0$ for every $t \in [0,\infty)$, then the function $x : [0,\infty) \to \mathbb{R}^n$ defined by $x(t) = 0$ for $t \in [0,\infty)$ is such that $x \in S(0)$.

Example 2.3. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t,x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases}$$

If $x : \mathbb{R} \to \mathbb{R}$ is given by

$$x(t) = \begin{cases} t+1, & t \leq -1 \\ 0, & -1 < t \leq 1 \\ t-1, & t > 1 \end{cases}$$

then x is a Carathéodory solution to Eq. (1) on $\mathbb{R}$. Since $\dot{x}(t) = f(t,x(t)) = 1$ for $t < -1$, $\dot{x}(t) = f(t,x(t)) = 0$ for $-1 < t < 1$, and $\dot{x}(t) = f(t,x(t)) = 1$ for $t > 1$.

Example 2.4. Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t,x) = \begin{cases} -x + \frac{1}{2}, & x < -\frac{1}{2} \\ x^2, & x \geq -\frac{1}{2}. \end{cases}$$

Then $x : \mathbb{R} \to \mathbb{R}$ defined by

$$x(t) = \begin{cases} \frac{1}{2} - e^{-t}, & t < 0 \\ \frac{-1}{t+2}, & t \geq 0 \end{cases}$$

is such that $\dot{x}(t) = f(t,x(t)) = -x(t) + \frac{1}{2}$ for $t < 0$, and $\dot{x}(t) = f(t,x(t)) = x^2(t)$ for $t > 0$. Thus, x is a Carathéodory solution to Eq. (1) on $\mathbb{R}$.

Example 2.5. Consider the function $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(t,x,y) = \begin{cases} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ (0,0), & (x,y) = (0,0). \end{cases}$$

Take an arbitrary $\theta \in [0, 2\pi)$. Let $X : [0, +\infty) \to \mathbb{R}^2$ be defined by

$$X(t) = (t\cos(\theta), t\sin(\theta)).$$

We have $\dot{X}(t) = f(t, X(t)) = (\cos(\theta), \sin(\theta))$ for $t > 0$ and hence $X \in S\big((0,0)\big)$.

2.3 Extension of solutions

Below we consider the maximal interval of existence of ordinary differential equations. For this, let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open connected set and $f_1 : \Omega \to \mathbb{R}^n$ a continuous function. If $(t_0, x_0) \in \Omega$, consider the initial value problem

$$x'(t) = f_1(t, x(t)), x(t_0) = x_0 \tag{2}$$

where $x(t)$ is a C^1 function from some interval $I \subset \mathbb{R}$ containing the initial time t_0 into $\mathbb{R}^n$. Then the initial value problem (2) has a unique local solution.

Theorem 2.6 ([12]). *For every $(t_0, x_0) \in \Omega$ the solution to the initial value problem (2) extends to a maximal existence interval $I = (\alpha, \beta)$. Furthermore, if $K \subset \Omega$ is any compact set containing the point (t_0, x_0), then there exist times $\alpha(K)$ and $\beta(K)$ such that $\alpha < \alpha(K) < \beta(K) < \beta$ and $(t, x(t)) \in \Omega \setminus K$, for $t \in (\alpha, \beta) \setminus [\alpha(K), \beta(K)]$.*

Example 2.7. Consider the function $f_1 : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ given by

$$f_1(t,x) = \frac{\arctan(\sqrt{x})}{\sqrt{x}}.$$

If $x_0 > 0$ and $t_0 = 0$, let $x(t)$ be the solution of the initial value problem (2) for $t \in (a, b)$. For each $t \in (0, b)$ we have

$$\begin{aligned} |x(t)| &= \left| x_0 + \int_0^t x'(s)ds \right| \\ &\leq |x_0| + \int_0^t |x'(s)|ds \\ &= |x_0| + \int_0^t \frac{\arctan(\sqrt{x(s)})}{\sqrt{x(s)}} ds \\ &\leq |x_0| + t. \end{aligned}$$

In a similar way, for each $t \in (a, 0)$ we have

$$|x(t)| \leq |x_0| + |t|.$$

It follows from Theorem 2.6 that $I = (-\infty, +\infty)$ is the maximal existence interval.

Example 2.8. Consider the set $A = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $f_1 : \mathbb{R} \times A \to \mathbb{R}^2$ be defined as

$$f_1(t, x, y) = \left(\arctan(x^2 + y^2), \frac{\arctan(x^2 + y^2)}{x^2 + y^2} \right).$$

Take $(x_0, y_0) \in A$ such that $x_0 > 0$ and $y_0 > 0$. If $t_0 = 0$, let $(x(t), y(t))$ denote the solution of the initial value problem (2) for $t \in (a, b)$. For each $t \in (0, b)$,

$$\begin{aligned}
\|(x(t), y(t))\| &= \left\| (x_0, y_0) + \int_0^t (x'(s), y'(s)) ds \right\| \\
&\leq \|(x_0, y_0)\| + \int_0^t \left\| (x'(s), y'(s)) \right\| ds \\
&\leq \|(x_0, y_0)\| + \int_0^t \Big(|x'(s)| + |y'(s)| \Big) ds \\
&= \|(x_0, y_0)\| + \int_0^t \arctan(x^2(s) + y^2(s)) ds \\
&\quad + \int_0^t \frac{\arctan(x^2(s) + y^2(s))}{x^2(s) + y^2(s)} ds \\
&\leq \|(x_0, y_0)\| + \int_0^t \frac{\pi}{2} ds + \int_0^t ds \\
&= \|(x_0, y_0)\| + (\frac{\pi}{2} + 1)t.
\end{aligned}$$

Similarly, for each $t \in (a, 0)$ we have

$$\|(x(t), y(t))\| \leq \|(x_0, y_0)\| + (\frac{\pi}{2} + 1)|t|$$

and from Theorem 2.6 it follows that $I = (-\infty, +\infty)$ is the maximal existence interval.

3 LYAPUNOV INSTABILITY

The main results of the manuscript are stated in Theorems 3.4 and 3.6. Theorem 3.4 is established from Lyapunov's first instability theorem (see [7, Theorem 9.16]). On the other hand, Theorem 3.6 is established from Chetaev's instability theorem (see [7, Theorem 9.22]). The functions V in Theorems 3.4 and 3.6 are analogous to Lyapunov functions for ordinary differential equations (see, for example, [13]).

The Lyapunov stability of equilibrium point $x = 0$ of Eq. (1) can be formulated analogously to ordinary differential equations (see, for instance, [7], [12] and [14]).

In what follows, we have a concept of stability (in the sense of Lyapunov) to solution $x \equiv 0$

of Eq. (1) with initial condition $x(0) = x_0$.

Definition 3.1. The equilibrium point $x = 0$ of Eq. (1) is stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $\|x_0\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t \geq 0$, with $x \in S(x_0)$.

Definition 3.2. The equilibrium point $x = 0$ of Eq. (1) is unstable if it is not stable. In this case, there exist $\varepsilon > 0$ and sequences $\{x_m\}$ and $\{t_m\}$ such that $\|\phi_m(t_m)\| \geq \varepsilon$ for all m, whenever $\phi_m \in S(x_m)$.

Definition 3.3. It is said that a continuous function $\psi : [0, r_1] \to [0, \infty)$ (respectively, $\psi : [0, \infty) \to [0, \infty)$) belongs to class $\mathcal{K}$ ($\psi \in \mathcal{K}$), if $\psi(0) = 0$ and if ψ is strictly increasing on $[0, r_1]$ (respectively, on $[0, \infty)$).

Below, $B(h)$ denotes the open ball of radius h centered at origin, that is,

$$B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}.$$

Theorem 3.4. *Consider $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz continuous. Suppose that in every neighborhood of the origin there are points x such that $V(0, x) > 0$. Suppose, moreover, that there exist functions $\psi_1, \psi_2 \in \mathcal{K}$ obeying the following assertions:*

(i) for some $h > 0$, $|V(t, x)| \leq \psi_1(\|x\|)$ for all $(t, x) \in [0, \infty) \times B(h)$;

(ii) $\frac{d}{dt}V(t, \phi(t)) \geq \psi_2(\|\phi(t)\|)$ for a.e. $t \in [0, \infty)$, and for all $\phi \in S(x_0)$ with $\|x_0\| < h$.

Then the equilibrium point $x = 0$ of Eq. (1) is unstable.

Proof. Let $\varepsilon > 0$ be such that $\varepsilon \leq h$. Take a sequence of points $\{x_m\}$ such that $0 < \|x_m\| < \varepsilon$, $V(0, x_m) > 0$ and $x_m \to 0$. Consider $\phi_m \in S(x_m)$ and $w_m : [0, \infty) \to \mathbb{R}$, with $w_m(t) = V(t, \phi_m(t))$. Since w_m is absolutely continuous,

$$w_m(t) - w_m(0) = \int_0^t \frac{d}{ds} w_m(s) ds$$

for all $t \in [0, \infty)$. Hence $\|\phi_m(t_m)\| = \varepsilon$ for some $t_m \in (0, \infty)$. For otherwise,

$$\begin{aligned}
\psi_1(\|\phi_m(t)\|) &\geq V(t, \phi_m(t)) = w_m(t) \\
&= w_m(0) + \int_0^t \frac{d}{ds} w_m(s) ds \\
&= w_m(0) + \int_0^t \frac{d}{ds} V(s, \phi_m(s)) ds \\
&\geq w_m(0) + \int_0^t \psi_2(\|\phi_m(s)\|) ds \\
&\geq w_m(0)
\end{aligned}$$

and then

$$\|\phi_m(t)\| \geq \psi_1^{-1}(w_m(0)) = \alpha_m > 0$$

for all $t \in [0, \infty)$. Thence

$$\begin{aligned}\psi_1(\varepsilon) > \psi_1(\|\phi_m(t)\|) &\geq w_m(t)\\ &\geq w_m(0) + \int_0^t \psi_2(\|\phi_m(s)\|)ds\\ &\geq w_m(0) + \int_0^t \psi_2(\alpha_m)ds\\ &= w_m(0) + t\psi_2(\alpha_m)\end{aligned}$$

for all $t \in [0, \infty)$. Taking $t \to \infty$ we get a contradiction. Therefore the equilibrium point $x = 0$ of Eq. (1) is unstable. □

Example 3.5. Let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(t,x,y) = \begin{cases} \left(\arctan(x^2+y^2), \frac{\arctan(x^2+y^2)}{x^2+y^2}\right), & y > 0\\ (0,0), & y = 0\\ \left(e^{-x^2-y^2}, \cos(\frac{1}{x^2+y^2})\right), & y < 0. \end{cases}$$

Let $V(t,x,y) = x$. Consider a sequence of points $\{(a_m,b_m)\}$ such that $a_m, b_m > 0$, $\|(a_m,b_m)\| < \varepsilon$ and $(a_m,b_m) \to (0,0)$. If $\phi_m = (x_m,y_m)$ is such that $\phi_m \in S\big((a_m,b_m)\big)$, it follows that

$$\frac{d}{dt}\phi_m(t) = \left(\arctan(x_m^2(t)+y_m^2(t)), \frac{\arctan(x_m^2(t)+y_m^2(t))}{x_m^2(t)+y_m^2(t)}\right)$$

and then $x_m(t), y_m(t) > 0$ for $t \geq 0$. Thus,

$$\begin{aligned}\frac{d}{dt}V(t,\phi_m(t)) &= \frac{d}{dt}x_m(t)\\ &= \arctan(x_m^2(t)+y_m^2(t))\\ &= \psi(\|\phi_m(t)\|)\end{aligned}$$

for a.e. $t \in [0,\infty)$, with $\psi(u) = \arctan(u^2)$ for all $u \geq 0$. From Theorem 3.4 and its proof we conclude that the equilibrium point $(x,y) = (0,0)$ of Eq. (1) is unstable.

Theorem 3.6. *Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous. Suppose that V satisfies the following properties:*

(i) for each $\varepsilon > 0$ and for any $t \geq 0$, there exists $\overline{x} \in B(\varepsilon)$ such that $V(t,\overline{x}) < 0$. Consider a fixed constant $h > 0$. The set $\{(t,x) \in [0,\infty) \times B(h) : V(t,x) < 0\}$ is called

the "domain $V < 0$". The domain $V < 0$ is bounded by the hypersurfaces which are determined by $\|x\| = h$ and by $V(t,x) = 0$ and it may consist of several component domains;

(ii) there is a component domain D of the domain $V < 0$ such that V is bounded below and $0 \in \partial D$;

(iii) there exists $\psi \in \mathcal{K}$ such that $\frac{d}{dt}V(t,\phi(t)) \leq -\psi(|V(t,\phi(t))|)$ for a.e. $t \in [0,\infty)$, and for all $\phi \in S(x_0)$ with $(0,x_0) \in D$ and $x_0 \in B(h)$.

Then the equilibrium point $x = 0$ of Eq. (1) is unstable.

Proof. Let $M > 0$ be such that $-M \leq V(t,x)$ for all $(t,x) \in D$. Take $\delta > 0$ arbitrary and consider $(0,x_0) \in D$, with $x_0 \in B(\delta) \cap B(h)$. If $\phi_0 \in S(x_0)$, then the function $w : [0,\infty) \to \mathbb{R}$ given by $w(t) = V(t,\phi_0(t))$ is absolutely continuous. For each $t \geq 0$ we have $|w(t)| \geq |w(0)|$, since

$$
\begin{aligned}
w(t) &= w(0) + \int_0^t \frac{d}{ds} w(s) ds \\
&\leq w(0) - \int_0^t \psi(|w(s)|) ds \\
&\leq w(0) < 0
\end{aligned}
$$

for all $t \geq 0$. Thus, $\|\phi_0(t_0)\| = h$ for some $t_0 \in (0,\infty)$. For otherwise,

$$
\begin{aligned}
w(t) &\leq w(0) - \int_0^t \psi(|w(s)|) ds \\
&\leq w(0) - \int_0^t \psi(|w(0)|) ds \\
&= w(0) - t\psi(|w(0)|)
\end{aligned}
$$

and then $w(t) \to -\infty$ as $t \to +\infty$. What contradicts the lower bound $-M \leq w(t)$. Then there exists $t_0 > 0$ such that $(t_0, \phi_0(t_0)) \in \partial D$. Since $w(t_0) < 0$, we have $\|\phi_0(t_0)\| = h$. So the equilibrium point $x = 0$ of Eq. (1) is unstable. $\square$

Example 3.7. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$
f(t,x) = \begin{cases} \frac{\arctan(\sqrt{x})}{\sqrt{x}}, & x > 0 \\ 0, & x = 0 \\ \cos(\frac{1}{x^2}) + \sin(\frac{1}{x^3}), & x < 0. \end{cases}
$$

Consider the function $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ given by

$$
V(t,x) = \begin{cases} -x^2, & x \geq 0 \\ x^2, & x < 0. \end{cases}
$$

Take a constant $h > 0$. In this case the domain $V < 0$ is given by $[0, \infty) \times (0, h)$. Take $D = [0, \infty) \times (0, h)$ and let $\delta > 0$ be arbitrary. Consider $(0, x_0) \in D$, with $x_0 < \delta$. If $\phi_0 \in S(x_0)$ it follows that

$$\frac{d}{dt}\phi_0(t) = \frac{\arctan(\sqrt{\phi_0(t)})}{\sqrt{\phi_0(t)}}$$

and then $\phi_0(t) > 0$ for $t \geq 0$. Thus,

$$\begin{aligned} \frac{d}{dt}V(t, \phi_0(t)) &= -2\phi_0(t)\frac{d}{dt}\phi_0(t) \\ &= -2\sqrt{\phi_0(t)}\arctan(\sqrt{\phi_0(t)}) \\ &= -\psi(|V(t, \phi_0(t))|) \end{aligned}$$

for a.e. $t \in [0, \infty)$, with $\psi(u) = 2\sqrt[4]{u}\arctan(\sqrt[4]{u})$ for all $u \geq 0$. It follows from Theorem 3.6 and its proof that the equilibrium point $x = 0$ of Eq. (1) is unstable.

4 CONCLUSIONS

The manuscript contributes to the qualitative theory of discontinuous differential equations. More specifically, the manuscript establishes two results on Lyapunov instability from Lyapunov's second method for ordinary differential equations. The two results established here are stated in Theorems 3.4 and 3.6. The Theorems 3.4 and 3.6 are analogies of Lyapunov's first instability theorem and Chetaev's instability theorem, respectively.

REFERENCES

[1] A. Bressan, "Unique Solutions for a Class of Discontinuous Differential Equations", *Proceedings of the American Mathematical Society*, vol. 104, no. 3, pp. 772–778, 1988. https://doi.org/10.1090/S0002-9939-1988-0964856-0

[2] O. Hájek, "Discontinuous Differential Equations. I", *Journal of Differential Equations*, vol. 32, no. 2, pp. 149–170, 1979. https://doi.org/10.1016/0022-0396(79)90056-1

[3] A. Bacciotti and F. Ceragioli, "Nonpathological Lyapunov Functions and Discontinuous Carathéodory Systems", *Automatica*, vol. 42, no. 3, pp. 453–458, 2006. https://doi.org/10.1016/j.automatica.2005.10.014

[4] P. G. Stechlinski and P. I. Barton, "Dependence of Solutions of Nonsmooth Differential-Algebraic Equations on Parameters", *Journal of Differential Equations*, vol. 262, no. 3, pp. 2254–2285, 2017. https://doi.org/10.1016/j.jde.2016.10.041

[5] D. Shevitz and B. Paden, "Lyapunov Stability Theory of Nonsmooth Systems", *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1910–1914, 1994. https://doi.org/10.1109/9.317122

[6] M. Grzanek, A. Michalak, and A. Rogowski, "A Nonsmooth Lyapunov Function and Stability for ODE's of Carathéodory Type", *Nonlinear Analysis: Theory, Methods and Applications*, vol. 69, no. 1, pp. 337–342, 2008. https://doi.org/10.1016/j.na.2007.05.022

[7] R. K. Miller and A. N. Michel *Ordinary Differential Equations*. 1 ed. New York: Academic Press, 1982.

[8] J. K. Hale *Ordinary Differential Equations*. 2 ed. Huntington, N.Y.: Robert E. Krieger Publishing Co., 1980.

[9] I. P. Natanson *Theory of Functions of a Real Variable*. 1 ed. New York: Frederick Ungar Publishing Co., 1955.

[10] H. L. Royden *Real Analysis*. 1 ed. New York: The Macmillan Co., 1963.

[11] G. V. Smirnov *Introduction to the Theory of Differential Inclusions*. 1 ed. Providence, RI: American Mathematical Society, 2002.

[12] T. C. Sideris *Ordinary Differential Equations and Dynamical Systems*. 1 ed. Paris: Atlantis Press, 2013.

[13] W. Hahn *Stability of Motion*. 1 ed. Berlin: Springer, 1967.

[14] T. Yoshizawa *Stability Theory by Liapunov's Second Method*. 1 ed. Tokyo: The Mathematical Society of Japan, 1966.

BRIEF BIOGRAPHY

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