

Primeness in Quantales

1st Flaulles Boone Bergamaschi

Dept. de Ciências Exatas e Tecnológicas
Univ. Estadual do Sudoeste da Bahia
Vitoria da Conquista, Brasil
flaulles@yahoo.com.br

2nd Regivan H. N. Santiago

Dept. de Inf. e Matemática Aplicada
Univ. Federal do Rio Grande do Norte
Natal, Brasil
regivan@dimap.ufrn.br

3rd Aleksandra Oliveira Andrade

Dept. de Ciências Exatas e Tecnológicas
Univ. Estadual do Sudoeste da Bahia
Vitoria da Conquista, Brasil
aleksandra0andrade@gmail.com

Abstract—In this paper we propose a new concept of primeness in quantales. It is proved that this concept coincide with classical definition in commutative quantales, but no longer valid in the noncommutative setting. Also, the notions of strong and uniform strong primeness are investigated.

Keywords—Primeness; Prime ideal; Strong; Quantales.

I. INTRODUCTION

As it is known, ideals are the main object in the investigation of ring theory and provide important information about the rings because they are structural pieces. The same may occur in quantales. The definition of prime ideals proposed in [1] is based on elements of a quantale and we ponder it is geared to commutative environment. When we move from commutative to the noncommutative setting, elementwise should be replaced by an approach based on ideals. Nevertheless, some authors defined the concept of primeness for commutative and non-commutative cases without realizing that this concept may not be suitable for noncommutative setting as it was well shown by Navarro et. al. in [2]. We state that the concept of prime ideal of general quantales could be defined as it is done in ring theory, i.e. based on ideals.

The main aim of this paper is to study the notion of primeness in the following perspective: we rename prime ideal defined in [1] to *completely prime ideal* and define a new concept of prime ideal for general quantales. Then we translate an important result in ring theory for quantales environment (theorem 2) to prove that these two concepts coincide in the commutative setting, but are no longer valid in the noncommutative setting (see Proposition 3). Also, based on the studies of Lawrence and Handelman [3], started in 1975, the notion of strong primeness is investigated for general quantales.

II. BASIC ON QUANTALES

This section proposes a new concept of prime ideals suitable for commutative and non-commutative quantales. The definition of prime ideal used in [4] and [1] will be called herein completely prime ideals. We drew attention to the theorem 2 where prime ideals can be characterized in a certain way via elements. The Proposition 4 shows that in the commutative case, prime and completely prime concepts coincide, which are no longer valid in the noncommutative setting according to Proposition 3. Finally, the concept of quantale prime is proposed.

Definition 1: [5] A **quantale** is a complete lattice Q with an associative binary operation \circ satisfying:

$$a \circ \left(\bigvee_{k \in K} b_k \right) = \bigvee_{k \in K} (a \circ b_k), \quad \left(\bigvee_{k \in K} a_k \right) \circ b = \bigvee_{k \in K} (a_k \circ b)$$

for all $a, b, a_k, b_k \in Q$ and $k \in K$, where K is a index set.

A quantale Q is called commutative whenever $a \circ b = b \circ a$ for $a, b \in Q$. In this paper we denote the **least** and **greatest** elements of a quantale by \perp and \top respectively. If there exists an element e in Q such that $x \circ e = e \circ x = x$ for all x in Q the quantale is called a **quantale with identity**. In this paper we consider quantales with identity.

Definition 2: [4] Let Q be a quantale. A non-empty subset $I \subseteq Q$ is called a **right ideal** of Q if it satisfies the following conditions:

- i) $a, b \in I$ implies $a \vee b \in I$;
- ii) for all $a, b \in Q, a \in I$ and $b \leq a$ imply $b \in I$,
- iii) for all $x \in Q$ and $a \in I$, we have $a \circ x \in I$.

Similarly we may define **left ideal** replacing (iii) by: (iii') for all $x \in Q$ and $a \in I$, we have $x \circ a \in I$. If I is both right and left ideal of Q , we call I a **two-sided ideal** or simply an **ideal** of Q .

Clearly by (ii) $\perp \in I$. Also, the set of all ideals of Q is closed under arbitrary intersections.

In Q we denote the subset $I \circ J = \{i \circ j \in Q : i \in I \text{ and } j \in J\}$ and $A \vee B = \{a \vee b : a \in A \text{ and } b \in B\}$. Since the operation \circ is associative, we have $(A \circ B) \circ C = A \circ (B \circ C)$. Also, if A is an two-sided ideal, then $A \circ Q, Q \circ A, Q \circ A \circ Q \subseteq A$.

As usual, \vee induces an order relation \leq on Q by putting $x \leq y \Leftrightarrow x \vee y = y$. Moreover, \leq is a congruence i.e. for every $x, y, u, v \in Q$ if $x \leq y$ and $u \leq v$, then $x \circ u \leq y \circ v$. To prove this, we first observe that if $w \leq z$ then, for any $s \in Q$, $s \circ w \leq s \circ z$ and $w \circ s \leq z \circ s$ because $z = w \vee z$ implies $s \circ z = s \circ (w \vee z) = (s \circ w) \vee (s \circ z)$ and $z \circ s = (w \vee z) \circ s = (w \circ s) \vee (z \circ s)$; now suppose $x \leq y$ and $u \leq v$, then $x \circ u \leq y \circ u$ and $y \circ u \leq y \circ v$. Hence, $x \circ u \leq y \circ v$ by transitivity.

III. PRIMENESS

In what follows we propose a more general definition of prime ideals which encompasses commutative and non-commutative quantales.

Definition 3: A **prime ideal** in a quantale Q is any proper ideal P such that, whenever I, J are ideals of Q with $I \circ J \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

Definition 4: A subset P of a quantale Q is called **completely prime ideal** if x and y are two elements of Q such that their product $x \circ y \in I$, then $x \in I$ or $y \in I$.

As we will see the concept of prime and completely prime ideals are different. But in a commutative quantale this concept is the same.

Proposition 1: If P is completely prime, then P is prime.

Proof: Suppose that P is completely prime and $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. For all $i \in I$ we have $i \circ j \in I \circ J \subseteq P$, as P is completely prime and $j \notin P$, then $i \in P$. Therefore $I \subseteq P$. ■

The Proposition 3 will show that the converse of this Proposition is not true.

Definition 5: [4] Let Q be a quantale and $A \subseteq Q$. The least ideal containing A is called the ideal generated by A , and denoted as $\langle A \rangle$.

Clearly, $\langle \emptyset \rangle = \{\perp\}$. If $\emptyset \neq A \subseteq Q$, then we have the following result.

Proposition 2: [4] Let A be a non-empty subset of a quantale Q . Then $\langle A \rangle = \{x \in Q : x \leq \bigvee_{i=1}^n a_i, \text{ for some positive integer } n \text{ and } a_1, \dots, a_n \in A \cup (A \circ Q) \cup (Q \circ A) \cup (Q \circ A \circ Q)\}$.

We may denote $\langle a \rangle = \langle \{a\} \rangle$ and $a \circ Q = \{a\} \circ Q$.

Lemma 1: $\langle a \rangle \circ Q \subseteq \langle a \rangle$ for all $a \in Q$. If there exists an unit 1 in Q , then $\langle a \rangle \circ Q = \langle a \rangle$.

Proof: Let $x \circ q \in \langle a \rangle \circ Q$, where $x \in \langle a \rangle$ and $q \in Q$. Hence, $x \circ q \leq \bigvee_{i=1}^n (a_i \circ q)$, where $a_i \circ q \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$. Thus, $x \circ q \in \langle a \rangle$. On the other hand if there exists unit 1 in Q , we write $z \in \langle a \rangle$ as $z = z \circ 1$. Thus, $z \in \langle a \rangle \circ Q$ and we have $\langle a \rangle \circ Q = \langle a \rangle$. ■

Theorem 2: For an ideal P in Q the following statements are equivalent:

- (1) P is prime ideal;
- (2) $\langle a \rangle \circ \langle b \rangle \subseteq P$ implies $a \in P$ or $b \in P$;
- (3) $a \circ Q \circ b \subseteq P$ implies $a \in P$ or $b \in P$.

Proof: For (1) \Rightarrow (2) note that $\langle a \rangle$ and $\langle b \rangle$ are ideals of Q . As P is prime and $\langle a \rangle \circ \langle b \rangle \subseteq P$, then $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$. Hence, $a \in P$ or $b \in P$. For (2) \Rightarrow (1), assume that $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. Given $i \in I$ we have $\langle i \rangle \subseteq I$. Hence, $\langle i \rangle \circ \langle j \rangle \subseteq I \circ J \subseteq P$. By hypothesis $i \in P$ or $j \in P$. As $j \notin P$ then we have $i \in P$. Therefore, $I \subseteq P$.

For (3) \Rightarrow (1), assume that $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. Given $i \in I$ we have $i \circ Q \circ j \subseteq I \circ J \subseteq P$. Hence, $i \in P$ or $j \in P$, as $j \notin P$ then we have $i \in P$. Thus, $I \subseteq P$.

For (1) \Rightarrow (3), suppose $a \circ Q \circ b \subseteq P$, we first shall show that $\langle a \rangle \circ Q \circ \langle b \rangle \subseteq P$. For this, let $x \circ q \circ y \in \langle a \rangle \circ Q \circ \langle b \rangle$, where $x \in \langle a \rangle$, $y \in \langle b \rangle$ and $q \in Q$. Hence, by Proposition 2, $x \leq \bigvee_{i=1}^n a_i$ and $y \leq \bigvee_{j=1}^m b_j$, where $a_i \in (a \circ Q) \cup (Q \circ a) \cup (Q \circ a \circ Q)$ and $b_j \in (b \circ Q) \cup (Q \circ b) \cup (Q \circ b \circ Q)$. Hence $x \circ q \circ y \leq (\bigvee_{i=1}^n a_i) \circ q \circ (\bigvee_{j=1}^m b_j) = (\bigvee_{i=1}^n (a_i \circ q)) \circ (\bigvee_{j=1}^m b_j) = \bigvee_{i=1}^n (a_i \circ q \circ \bigvee_{j=1}^m b_j) = \bigvee_{i=1}^n (\bigvee_{j=1}^m (a_i \circ q \circ b_j))$.

Observe that $a_i \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$ and $b_j \in b \circ Q \cup Q \circ b \cup Q \circ b \circ Q$ it is no hard to see that $a_i \circ q \circ b_j \in a \circ Q \circ b \subseteq P$ for all i, j . As P is an ideal we have $x \circ q \circ y \in P$. Thus, $\langle a \rangle \circ Q \circ \langle b \rangle \subseteq P$. By the Lemma 1 $\langle a \rangle \circ Q = \langle a \rangle$. Then, $\langle a \rangle \circ Q \circ \langle b \rangle = \langle a \rangle \circ \langle b \rangle \subseteq P$. By the first proof ((1) \Leftrightarrow (2)) we have $a \in P$ or $b \in P$. ■

Proposition 3: There exists a noncommutative quantale where a prime ideal is not a completely prime ideal.

Proof:

Consider G the invertible 2×2 matrices under multiplication over the real interval $[0, 1]$ and the partial order $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$. According to Rosenthal [5], page 19, example 16, any complete partially ordered group (written multiplicatively) is a quantale with $a \circ b = a \cdot b$. Thus, G is a noncommutative quantale. Let $\langle 0 \rangle$ as an ideal generated by 0, clearly $\langle 0 \rangle = \{0\}$. We will show that the $\langle 0 \rangle$ (zero ideal) is prime, but not completely prime by using the Theorem 2 (3). Thus, suppose that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are two matrices such that $X \circ G \circ Y \subseteq \langle 0 \rangle$.

Hence $X \circ T \circ Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all matrix $T \in G$. Then, in particular,

$$X \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af \\ ce & cf \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } e = f = 0,$$

$$X \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ag & ah \\ cg & ch \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } g = h = 0,$$

$$X \circ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} be & bf \\ de & df \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } e = f = 0,$$

$$X \circ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} bg & bh \\ dg & dh \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } g = h = 0,$$

Hence, a solution must verify that $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $X \in \langle 0 \rangle$ or $Y \in \langle 0 \rangle$ and then $\langle 0 \rangle$ is prime. Nevertheless, $\langle 0 \rangle$ is not completely prime, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ although } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \langle 0 \rangle. \quad \blacksquare$$

Proposition 4: In a commutative quantale an ideal is completely prime iff it is prime.

Proof: If P is a completely prime ideal of a quantale Q , then by the Proposition 1 P is prime. On the other hand, suppose P is a prime ideal and $a \circ b \in P$ for any $a, b \in Q$. Let $x \circ y \in \langle a \rangle \circ \langle b \rangle$, where $x \in \langle a \rangle$ and $y \in \langle b \rangle$. Thus, $x \circ y \leq \bigvee_{i=1}^n a_i \circ \bigvee_{j=1}^m b_j = \bigvee_{i=1}^n \left(\bigvee_{j=1}^m (a_i \circ b_j) \right)$, where $a_i \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$ and $b_j \in b \circ Q \cup Q \circ b \cup Q \circ b \circ Q$. As Q is commutative $a \circ Q = Q \circ a = Q \circ a \circ Q$ and $b \circ Q = Q \circ b = Q \circ b \circ Q$. Thus, $a_i \circ b_j \in a \circ Q \circ b \circ Q = a \circ b \circ Q$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Hence, $a_i \circ b_j = a \circ b \circ q \in P$ and then $x \circ y \leq \bigvee_{i=1}^n \left(\bigvee_{j=1}^m (a_i \circ b_j) \right) \in P$. Therefore, $\langle a \rangle \circ \langle b \rangle \subseteq P$ and by the Theorem 2 we have $a \in P$ or $b \in P$. \blacksquare

In what follows, we will introduce the notion of quantale prime. As we know, in ring theory, a quantale is prime iff the ideal generated by 0 is a prime ideal. Then, the Proposition 5 translates this result into quantale environment.

Definition 6: A quantale Q is called **prime** if given $a, b \in Q$ with $a \neq \perp$ and $b \neq \perp$, there exists $f \in Q$ such that $a \circ f \circ b \neq \perp$.

Proposition 5: A quantale Q is prime iff $\langle \perp \rangle$ is a prime ideal.

Proof: Suppose Q prime and assume that $I \circ J \subseteq \langle \perp \rangle$, but $I, J \not\subseteq \langle \perp \rangle$, where I, J are ideals of Q . Thus, there exists $i \in I, j \in J$ such that $i, j \neq \perp$. As Q is prime, there exists $q \in Q$ such that $i \circ q \circ j \neq \perp$, then we have a contradiction because $i \circ q \circ j \in I \circ J \subseteq \langle \perp \rangle$. Hence, $I \subseteq \langle \perp \rangle$ or $J \subseteq \langle \perp \rangle$. On the other hand, suppose $\langle \perp \rangle$ is a prime ideal of Q . Given $a, b \neq \perp$ in Q , suppose $a \circ q \circ b = \perp$ for all $q \in Q$. Hence, $a \circ Q \circ b \subseteq \langle \perp \rangle$. As $\langle \perp \rangle$ is a prime ideal, then $a \in \langle \perp \rangle$ or $b \in \langle \perp \rangle$, but $a, b \neq \perp$. \blacksquare

IV. STRONG PRIMENESS IN QUANTALES

Strongly prime rings were introduced in 1973, as a prime ring with finite condition in the generalization of results on group rings proved by Lawrence in [3]. In this section we bring this concept to quantales making specific adaptations for this environment.

Definition 7: Let A be a subset of a quantale Q . The **right annihilator** of A is defined as $An_r(A) = \{x \in Q : Ax = \langle \perp \rangle\}$. Similarly, we can define the **left annihilator** An_l .

Definition 8: [3] A quantale Q is called **right strongly prime** if for each $x \in Q - \{\perp\}$ there exists a finite nonempty

subset F_x of Q such that $An_r(x \circ F_x) = \langle \perp \rangle$. Clearly if Q is right strongly prime, then Q is prime. The set F_x is called an insulator of x in Q .

Proposition 6: If Q is right strongly prime, then every nonzero ideal I of Q contains a finite subset F which has right annihilator zero.

Proof: Suppose Q right strongly prime. Let $x \in I$ and $x \neq \perp$ and $F = x \circ F_x \subseteq I$. Thus, $An_r(F) = \langle \perp \rangle$. \blacksquare

It is clear that every right strongly prime quantale is a prime quantale. It is also possible to define left strongly prime in a similar manner for right strong primeness.

Definition 9: A quantale Q is called **uniformly strongly prime (usp)** if the same right insulator may be chosen for each nonbottom element.

Proposition 7: A quantale Q is a right uniformly strongly prime iff there exists a finite subset $F \subseteq Q$ such that for any two nonbottom elements x and y of Q , there exists $f \in F$ such that $x \circ f \circ y \neq \perp$.

Proof: Let Q be uniformly right strongly prime quantale. Hence Q has a uniform right insulator F which is a finite set such that for any element $x \in Q$, $x \circ F$ has no nonbottom right annihilators. Thus, if x and y are any two nonbottom elements in Q , y cannot be in the annihilator of $x \circ F$. Hence there is an $f \in F$ such that $x \circ f \circ y \neq \perp$. For the reverse implication it is easy to see that if the condition is satisfied then for any $x \neq \perp$ in Q , no nonbottom element annihilates $x \circ F$ on the right. Hence Q is uniformly right strongly prime \blacksquare

It is clear that the condition in Proposition 7 is not one-sided; consequently, this condition is also equivalent to uniformly left strongly prime, and we have:

Corollary 3: Q is uniformly right strongly prime iff Q is uniformly left strongly prime.

Corollary 4: A quantale Q is uniformly strongly prime iff there exists a finite subset $F \subseteq Q$ such that $a \circ F \circ b = \perp$ implies $a = \perp$ or $b = \perp$ for all $a, b \in Q$.

Proof: Straightforward. \blacksquare

Definition 10: An ideal $P \neq \langle \perp \rangle$ of a quantale Q is called **uniformly strongly prime (usp)** ideal if there exists a finite subset $F \subseteq Q$ such that $a \circ F \circ b \subseteq P$ implies $a \in P$ or $b \in P$.

Proposition 8: An ideal I of a quantale Q is a usp ideal iff there exists a finite subset $F \subseteq Q$ such that for any two elements $a, b \in Q \setminus I$ (complement of I in Q), there exists $f \in F$ such that $x \circ f \circ y \notin I$.

Proof: Suppose I a usp ideal of Q . If $a \notin I$ and $b \notin I$ by Definition 10 $a \circ F \circ b$ is not a subset of I . Hence, there exists $f \in F$ such that $a \circ f \circ b \notin I$. For the converse, note that by hypothesis it is impossible to have $a \circ F \circ b \subseteq I$ and $a \notin I$ and $b \notin I$. \blacksquare

V. FINAL REMARKS

Prime ideals have developed an important role in ring theory and have attracted the attention of some researchers in the investigation of quantales. As prime ideals are structural pieces of a ring it is relevant to study its concept in order to establish a well-founded quantale theory. With this in mind, it is necessary to investigate primeness over arbitrary quantales, i.e. commutative and noncommutative setting. Therefore, this paper invites the reader to think about primeness in quantales bringing a new perspective for prime ideals.

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