Finitary ideals of direct products in quantales

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Abstract- In this paper, given two quantales non necessary with identity. We investigate the ideals, prime ideals, radical ideals, primary ideals, and maximal ideals of the direct product. Unlike the case where those quantale are unital, an ideal (or primary ideal, or maximal ideal) of the their direct product need not be a sub-product (Lemma 3.1) of ideals. The Theorem 4.2 extends the result on [3] **for the product of two quantales.**
Keywords- quantale, primness, maximality ideal

1 Introduction

The notion of quantale, which designates a complete lattice equipped with an associative binary multiplication distributing over arbitrary joins, appears in various areas of mathematics-in quantaloid theory, in non classical logic as completion of the Lindebaum algebra, and in different representations of the spectrum of a C^* algebra as many-valued and non commutative topologies. To put it briefly, its importance is no longer to be demonstrated. Quantales are ring-like structures in that they share with rings the common fact that while as rings are semi groups in the tensor category of abelian groups, so quantales are semi groups in the tensor category of sup-lattices. In 2008 Anderson and Kintsinger [1] characterized prime ideals, radical ideals, primary ideals and maximal ideals of $R \times S$ where R and S are commutative rings. In 2009, D. Anderson and V. Camillo [2] given exposition of Goursat's lemma which describes the sub groups of a direct product of two groups. A ring version giving the sub rings and ideals of a direct product of two rings is also given. In this paper we prove some results which are proved by Anderson and Kintsinger in reference [1]. Let us recall some definitions that exist in the literature.

[0]ReCiC-Revista de Ciéncia de Computao

2 BASIC ON QUANTALES

Definition 2.1. A quantale is a join complete lattice Q with an associative binary operation $\circ : Q \times Q \to Q$, called its multiplication, satisfying a distributive property such that for all elements x and y_i of Q, for all i in a set of indexed family I, we have the following identities: $x \circ (\bigvee y_i) = \bigvee (x \circ y_i)$ and $(\bigvee y_i) \circ x = \bigvee (y_i \circ x)$.

A quantale is unital if it has an identity element e for its multiplication: $x \circ e = x = e \circ x$, for all x in Q. In this case, the quantale is naturally a monoid with respect to its multiplication.

The largest element of Q is denoted by $\top$ and the smallest by $\bot$. An element a is called left-sided if $\top \circ a \leq a$, and it called right sided if $a \circ \top \leq a$ and two-sided if it is left-sided and right-sided. A commutative quantale is a quantale whose multiplication is commutative.

Given an ordered set $(Q, \leq)$ the downward closure of an element x, denoted by $\downarrow x$ is defined by: $\downarrow x = \{l \in Q : l \leq x\}$. If Q_1 and Q_2 are two quantales, then the binary product $Q_1 \times Q_2$ is a quantale where the supremum and the multiplication are defined component by component.

Definition 2.2. [3] A subset I of a quantale Q is called a finitary left ideal of the quantale Q provided that the following conditions are satisfy.

1. For all elements a and b of I, the supremum $a \vee b$ is an element of I.
2. For all element x of I and y an element of Q where $y \leq x$, we have y is an element of I.
3. For all element x of I and all element q of Q, we have $q \circ x$ is an element of I.

Right ideal is defined in a similar way, replace 3) by 3')

3' : For all element x of I and all element q of Q, we have $x \circ q$ is an element of I.

The ideal I is a two-sided ideal or, simply, an ideal if it is both a left and a right ideal.

The definition of ideal that we use in this paper is that used by Bergamaschi and Santiago (see [3]) and the paper of Shaoul Liang (see [7]). In all our work, ideal finitaire will be replaced by ideal itself.

As we work only finitaries ideals, they will simply be called ideals, left ideals or right ideals.

For all quantales Q_1 and Q_2, non necessary with identity, we investigate the ideals, prime ideals, primary ideals, and maximal ideals of the quantale $Q_1 \times Q_2$. Unlike the case where the quantales Q_1 and Q_2 have are unitary, an ideal (or primary ideal, or maximal ideal) of $Q_1 \times Q_2$ need not be a 'sub product' $I \times J$ of ideals. We show that for a quantale Q_1, for each commutative quantale Q_2 every ideal (or primary ideal, or maximal ideal) is a sub-product if and only if Q is an e-quantale (that is, for $q \in Q$, there exists an element $e_q \in Q$ satisfy the identity $e_q \circ q = q$).

An arbitrary intersection of left ideals (resp right ideals, resp. ideal) of a quantale is again a left ideal (resp. right ideal, resp. ideal).

It should observed that in this case, $\top \circ \bigvee S$ is a left-sided element where S is a subset of a quantale Q. Thus the following result.

Lemma 2.3. For any left-sided element x of a quantale Q, the down-set $\downarrow x$ is an ideal with biggest element x.

Proof. Let x be an left-sided element of Q; the subset $\downarrow x$ is closed under binary supremum and it is closed immediate. Now if q and l are elements of Q and $l \leq x$, then $q \circ l \leq q \circ x \leq \top \circ x \leq x$. The last inequality comes from that x is left-sided element. □

Given a subset S of a quantale Q, the least left ideal containing S, which we denote by $\langle S \rangle_l$ is called the left ideal generated by S. In particular the left ideal generated by a empty set is $\{\bot\}$ where $\bot$ is the least element of Q. We also denote by $\langle S \rangle_r$ (resp. $\langle S \rangle$) the right ideal (resp. ideal) generated by S.

If K and J are two sub-sets of a quantale Q, we define the subset $K \circ J$ by the set $K \circ J := \{k \circ j : k \in K, j \in J\}$ and the subset $K \bigvee J$ by the following set: $K \bigvee J := \{k \bigvee j : k \in K, j \in J\}$.

The next lemma is proved by Qingjun Luo and Guojun Wang.

Lemma 2.4. [6] For any quantale Q the following properties hold.

(i) For any subset S of Q; the ideal generated by S is given by the set $\langle S \rangle = \{x \in Q : x \leq \bigvee_{i=1}^{n} a_i$ for some positive integer n, $a_i \in S \cup (Q \circ S) \cup (S \circ Q) \cup (Q \circ S \circ Q)\}$. In particular, for each element a of Q, $\langle a \rangle = \{x \in Q : x \leq \bigvee_{i=1}^{n} a_i$ for all positive integer n, $a_i \in \{a\} \cup (Q \circ a) \cup (a \circ Q) \cup (Q \circ a \circ Q)\}$.

(ii) The ideals of Q are precisely the sets $\langle a \rangle$ with a be an element of Q satisfies $\bigvee_{i=1}^{n} a_i \leq a$ for some positive integer n, $a_i \in \{a\} \cup (Q \circ a) \cup (a \circ Q) \cup (Q \circ a \circ Q)$.

The proof of the next lemma is similar, we state it.

Lemma 2.5. Let S be a non empty subset of a quantale Q.
(i) The left ideal of Q generated by S is given by $\langle S \rangle_l = \{x \in Q : x \leq \bigvee_{i=1}^{n} a_i$ for some positive integer n, $a_i \in S \cup (Q \circ S) \cup (Q \circ S \circ Q)\}$.
(ii) The right ideal of Q generated by S is given by $\langle S \rangle_r = \{x \in Q : x \leq \bigvee_{i=1}^{n} a_i$ for some positive integer n, $a_i \in S \cup (S \circ Q) \cup (Q \circ S \circ Q)\}$.

Lemma 2.6. [3] For all element a of a quantale Q, the following inclusion hols: $\langle a \rangle \circ Q \subseteq \langle a \rangle$. If there exists an unit e in Q, then $\langle a \rangle \circ Q = \langle a \rangle$.

Definition 2.7.
- A proper ideal M of a quantale Q is said to be maximal if for each ideal K of Q such that the inclusions $M \subseteq K \subsetneq Q$ are holds, we have the equality $K = M$.
- An ideal J of a quantale Q is said to be completely prime (respectively completely semi-prime) if for each elements x and y of Q, the element $x \circ y$ is in J implies that x is and element of J or y is an element of J (respectively the element $x \circ x$ is an element of J implies that x is an element of J).
- An ideal P in a quantale Q is said to be prime (respectively semi-prime) if P for each Ideals K and J of Q, the ideal $K \circ J$ satisfy $K \circ J \subseteq P$ implies that K is a subset of P or J is a subset of P (respectively the ideal $K \circ K$ is a subset of P implies that K is a subset of P).

It is proved in reference [3] that every completely prime ideal of a quantale Q is prime, and if Q is commutative quantale, we have the equivalence of the two notions.

If Q_1 and Q_2 are two quantales, a morphism from Q_1 to Q_2 is an arrow f defined from Q_1 to Q_2 such that $f(x \circ y) = f(x) \circ f(y)$ and $f(\bigvee x_i) = \bigvee f(x_i)$, for all elements x, y and x_i of Q, where i is element of an indexed family I. The collection of quantales and morphisms of quantales form a category, denoted by *Quant*.

For example, if R is a ring, let $Id(R)$ the collection of all left ideals of R; $Id(R)$ has a structure of quantale where the operations $\bigvee$ and multiplication $\circ$ are defined as follows. $\bigvee K_i = \langle \cup K_i \rangle$, the ideal generated by the union of the family $\{K_i : i \in I\}$ and $K \circ J = \{k.j$, where $k \in K$ and $j \in J\}$. Let Q_3 the set of three elements $\{0, a, 1\}$ with order $0 \leq a \leq 1$;

Q is a quantale where the multiplication is given by the following table.

$\circ$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Example 2.8. Let $\mathcal{M} = (M, *, e)$ be a monoid; the power set $\mathcal{P}(M)$ has the structure of quantale where the suprema is given by the union and the multiplication is defined as follows: for each subsets A and B of M, $A \circ B$ is the set given by $A \circ B = \{a * b : a \in A, b \in B\}$. The identity in this quantale is the subset $\{e\}$. It is clear that if the monoid $\mathcal{M}$ is commutative, then again the quantale $(\mathcal{P}(M), \subseteq, \circ)$ is again commutative. In reference [4] the author call the quantale $(\mathcal{P}(M), \subseteq, \circ)$the relational quantale. It is proved in reference [4] that each quantale is isomorphic to a relational quantale.

3 Ideals of product of two quantales

In a quantale Q, it is away that for all element q of Q, we have $q \circ \bot = \bot \circ q = \bot$; the least element $\bot$ is the empty suprema, so $q \circ \bot = q \circ \bigvee \emptyset = \bigvee_{x \in \emptyset}(q \circ x) = \bigvee \emptyset = \bot$ We next turn to direct product of quantales and their ideals. If Q_1 and Q_2 are quantales, and I_1 and I_2 are ideals of Q_1 and Q_2 respectively, then $I_1 \times I_2$ is an ideal of the quantale $Q_1 \times Q_2$. Similar statements hold for right and left ideal ideals. If Q_1 and Q_2 have an identity, then every ideal (right, left or two-sided) has this form. This is our next lemma.

Lemma 3.1. For any unital quantales Q_1 and Q_2, every left ideal of the quantale $Q_1 \times Q_2$ is a sub-product of left ideals, that is the ideal of $Q_1 \times Q_2$ has the form $K_1 \times K_2$, where K_i is a left ideal of Q_i, for $i = 1$ or $i = 2$.

Proof. First, if K_i is an ideal of Q_i, then $K_1 \times K_2$ is clearly an ideal of $Q_1 \times Q_2$. Secondly, let K be an ideal of the quantale $Q_1 \times Q_2$; we will prove that K is a sub-product. Put $K_1 = \{a \in Q_1 : (a, \bot) \in K\}$ and $K_2 = \{b \in Q_2 : (\bot, b) \in K\}$. We have $(\bot, \bot) \in K$, then $\bot \in K_1$. Now, let $a \in K_1$ and $a' \in Q_1$; We have $(a', \bot) \circ (a, \bot) = (a' \circ a, \bot)$ is an element of K, because $(a, \bot) \in K$ and K is an ideal of $Q_1 \times Q_2$. So $a \circ a'$ is an element of K_1. Let a_1, and a_2 be two elements of K_1; $(a_i, \bot)$ are in K for $i = 1, 2$. Therefore the equality $(a_1, \bot) \vee (a_2, \bot) = (a_1 \vee a_2, \bot)$ holds and we conclude that $(a_1, \bot) \vee (a_2, \bot)$ is an element of K; this implies that $a_1 \vee a_2$ is an element of K_1. Let x_1 be an element of K_1 and let y_1 be an element of Q_1 with respect $y_1 \leq x_1$. The element $(x_1, \bot)$ is in K; so we have $(y_1, \bot) \leq (x_1, \bot)$ and then $(y_1, \bot)$ is an element of K; so y_1 is an element of K_1 and we conclude that K_1 is an ideal; similarly, K_2 is an ideal of Q_2. To finish this proof, it is let to check that is $K = K_1 \times K_2$. Let (a, b) an element of K; then $(a, \bot) = (e, \bot) \circ (a, b)$ is an element of K and a is an element of K_1. Similar we prove that b is an element of K_2. Now, for all element (a, b) of the set $K_1 \times K_2$, $(a, \bot)$ and $(\bot, b)$ are elements of K_1 and K_2 respectively; we have the following equality: $(a, b) = (e, \bot) \circ (a, \bot) \vee (\bot, e) \circ (\bot, b)$, and we conclude that (a, b) is an element of K. We conclude that $K = K_1 \times K_2$. ☐

Proposition 3.2. Let Q_1 and Q_2 be two quantales with least elements denoted by $\bot$; the following conditions are equivalent .

1. Every ideal K of $Q_1 \times Q_2$ is a sub-product.
2. For each element a of Q_1 and each element b of Q_2, we have the equality $\langle (a, b) \rangle = \langle a \rangle \times \langle b \rangle$.
3. For each element a of Q_1 and each element b of Q_2, the element $(a, \bot), (\bot, b)$ is an element of $\langle (a, b) \rangle$.

Proof. (1) $\Rightarrow$ (2). Assume that every ideal of the quantale $Q_1 \times Q_2$ is a sub-product; let a be an element of Q_1 and b be an element of Q_2. We want to show the equality $\langle (a, b) \rangle = \langle a \rangle \times \langle b \rangle$. The subset $\langle (a, b) \rangle$ is an ideal of the quantale $Q_1 \times Q_2$, then by hypothesis it is the form $K_1 \times K_2$ where K_i is an ideal of Q_i, $i = 1, 2$; the pair (a, b) is an element of the set $K_1 \times K_2$, then a is an element of K_1 and b is an element of K_2, this implies the following inclusions $\langle a \rangle \subseteq K_1$ and $\langle b \rangle \subseteq K_2$. So the inclusion $\langle a \rangle \times \langle b \rangle \subseteq \langle (a, b) \rangle$ hols. The pair (a, b) is an element of $\langle a \rangle \times \langle b \rangle$ and $\langle a \rangle \times \langle b \rangle$ is an ideal of $Q_1 \times Q_2$, then $\langle (a, b) \rangle \subseteq \langle a \rangle \times \langle b \rangle$, therefore $\langle a \rangle \times \langle b \rangle = \langle (a, b) \rangle$.
(2) $\Rightarrow$ (3). Assume that for each element a of Q_1 and b an element of Q_2, the pair is write $\langle (a, b) \rangle = \langle a \rangle \times \langle b \rangle$. Let a be an element of Q_1 and b be an element of Q_2; The pair (a, b) is an element of $\langle (a, b) \rangle = \langle a \rangle \times \langle b \rangle$, so $(a, \bot)$ is an element of $\langle a \rangle \times \langle b \rangle$; therefore the pair $(a, \bot)$ is an element of $\langle (a, b) \rangle$.
(3) $\Rightarrow$ (1). Assume that for each element a of the quantale Q_1 and each element b of the quantale Q_2, the pairs $(a, \bot)$ and $(\bot, b)$ are elements of $\langle (a, b) \rangle$. Let K be an ideal of the quantale $Q_1 \times Q_2$; put $K_1 := \{a \in Q_1 : (a, \bot) \in K\}$ and $K_2 := \{b \in Q_2 : (\bot, b) \in K\}$. Let a be an element of K_1 and a' be an element of Q_1 such that $a' \leq a$; the pair $(a, \bot)$ is an element of K and $(a', \bot) \leq (a, \bot)$, then the pair $(a', \bot)$ is an element of K and a' is an element of K_1. For all pair (a, b) of $K_1 \times K_2$, a is an element of K_1

and b is an element of K_2, then the pairs $(a, \perp)$ and $(\perp, b)$ are elements of K and the following equality hols: $(a, b) = (a, \perp) \vee (\perp, b)$ and conclude that the pair (a, b) is an element of K; this prove that the set $K_1 \times K_2$ is a subset of K. To finish, let (a, b) be an element of K; the pairs $(a, \perp)$ and $(\perp, b)$ are elements of $\langle(a, b)\rangle$, so a is element of K_1 and b is element of K_2 and we conclude that the equality $K = K_1 \times K_2$ holds. □

Of course Lemma 3.1 may fail if Q_1 and Q_2 do not have an identity. For example, we next give a partial converse to Lemma 3.1. Let us call a quantale a left e-quantale if for each element q of Q there exists an element e_q of Q depending of q with $e_q \circ q = q$. Note that Q is a left e-quantale if and only if $Q \circ I = I$ for each left ideal I of Q.

Theorem 3.3. For a quantale Q_1, with top element $\top$ and bottom element $\perp$, satisfies for all elements a and b of Q_1, $a \leq b \circ a$ implies that $a = b \circ a$. The following conditions are equivalent in Q_1.
(1) The quantale Q is a left e-quantale (that is, for each element q of Q, there exists an element e_q of Q_1 with $e_q \circ q = q$).
(2) For each unital quantale Q_2, each left ideal of $Q_1 \times Q_2$ is a sub-product.
(3) For all natural number n such that $n \geq 2$, each left ideal of the quantale Q_1^n has the form $K_1 \times ... \times K_n$ where K_i is an left ideal of Q_1 for all i element of $\{1, 2, ..., n\}$.
(4) Every left ideal of the quantale $Q_1 \times Q_1$ is a sub-product.

Proof. (1) $\Rightarrow$ (2). Assume that Q_1 is an e-quantale and let Q_2 be an unital unital quantale denoted e. Let K be an ideal of $Q_1 \times Q_2$. Put $K_1 = \{a \in Q_1 : (a, \perp) \in K\}$ and $K_2 = \{b \in Q_2 : (\perp, b) \in K\}$; For each pair (a, b) of $K_1 \times K_2$, a is an element of K_1 and b is an element K_2, so $(a, \perp)$ is an element of K and $(\perp, b)$ is an element of K; $(a, b) = (a, \perp) \vee (\perp, b)$ is an element of K and $K_1 \times K_2 \subseteq K$. Now let (a, b) be element of K; choose e_a, element of Q_1 with $e_a \circ a = a$. Then $(a, \perp) = (e_a, \perp) \circ (a, b)$ is an element of K; so a is an element of K_1. And $(\perp, b) = (\perp, e) \circ (a, b)$ is an element of K; so b is an element of K_2.
(2) $\Rightarrow$ (3). By induction on n, if $n = 2$, apply (2) with $Q_2 = Q_1$. Assume the result for $n - 1$ and take Q' given by $Q_2 = Q_1^{n-1}$ and conclude.
(3) $\Rightarrow$ (4): clear.
(4) $\Rightarrow$ (1). Let q be an element of Q, then the pair (q, q) is an element of $Q_1 \times Q_1$ and the pair (q, q) is an element of $\langle(q, q)\rangle = \langle q\rangle \times \langle q\rangle$. So the pair $(q, \perp)$ is an element of $\langle q\rangle \times \langle q\rangle$ (use 3.2); there exists a finite family $\{a_i : 1 \leq i \leq n\}$ of elements of $Q \circ q \cup Q \circ q$, where n is a positive entire number, such that $q \leq \bigvee_{i=1}^{n} a_i$; so $q \leq (\bigvee_{i=1}^{n} b_i) \circ q$. We conclude that $q = e_q \circ q$, where $e_q = \bigvee_{i=1}^{n} b_i$. □

Example 3.4. If Q_1 is a idempotent quantale (That is $x \circ x = x$ for all element x of Q_1), then Q is an e-quantale. Every left ideal of $Q_1 \times Q_2$ is a sub-product, for all unital quantale Q_2.

Example 3.5. Let $Q = \{0, a, b, c, 1\}$ with the order $\leq$, given by $0 \leq a \leq c \leq 1$ and $0 \leq b \leq c \leq 1$ is a quantale where their multiplication is giving by the following table.

$\circ$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	1	a	b	c	1

We see that Q_1 is an e-quantale, and any (left, right, two-sided) ideal of $Q_1 \times Q_2$, where Q_2 is a commutative quantale is a sub-product.

We leave it to the reader to define a right e-quantale and to state versions of theorem 3.3 for right ideals and two-sided ideals. Note that if $\{Q_i : i \in I\}$ is any nonempty family of left e-quantales, then their direct product $\prod_{i \in I} Q_i$ with coordinate-wise operations is again a left e-quantale. In particular, an infinite direct product of quantales each having an identity is both a left and right e-quantale, but does not have an identity.

4 Primeness

Lemma 4.1. [3] For an ideal P in a quantale Q with unity, the following statements are equivalent:
(1) the ideal P is prime.
(2) For any elements a and b of Q, the inclusion $\langle a\rangle \circ \langle b\rangle \subseteq P$ holds implies $\langle a\rangle \subseteq P$ or $\langle b\rangle \subseteq P$.
(3) For any elements a and b of Q, the inclusion $a \circ Q \circ b \subseteq P$ holds implies that a is an element of P or b is an element of P.

Theorem 4.2. Let Q_1 and Q_2 be quantales. Then an ideal $\mathcal{P}$ of the quantale $Q_1 \times Q_2$ is prime if and only if $\mathcal{P}$ has the form $P_1 \times Q_2$ where P_1 is a prime ideal of Q_1 or $Q_1 \times P_2$ where P_2 is a prime ideal of Q_2.

Proof. $\Rightarrow$) Assume that P_1 is a prime ideal of Q_1. Let I and J be two ideals of $Q_1 \times Q_2$ such that $I \circ J \subseteq P_1 \times Q_2$; I and J are ideals of the quantale $Q_1 \times Q_2$, then I and J have the form $I = I_1 \times I_2$ and $J = J_1 \times J_2$ where I_k and J_k are ideals of Q_k for $k = 1, 2$. The following inclusions are hold: $I \circ J = (I_1 \times I_2) \circ (J_1 \times J_2) = (I_1 \circ J_1) \times (I_2 \circ J_2) \subseteq P_1 \times Q_2$;

so $I_1 \circ J_1 \subseteq P_1$ and P_1 is a prime ideal. So $I_1 \subseteq P_1$ or $J_1 \subseteq P_1$ and deduce that $I \subseteq P_1 \times Q_2$ or $J \subseteq P_1 \times Q_2$ and conclude that $P_1 \times Q_2$ is a prime ideal. Similarly we prove that if P_2 is a prime ideal of Q_2, then $Q_1 \times P_2$ is a prime ideal of $Q_1 \times Q_2$. Conversely, Assume that $\mathcal{P}$ is a prime ideal of $Q_1 \times Q_2$. Denoted by $0 = \{\perp\}$ the least ideal of Q_1 or Q_2. Now $(0 \times Q_2) \circ (Q \times 0) \subseteq \mathcal{P}$, so either $0 \times Q_2 \subseteq \mathcal{P}$ or $Q_1 \times 0 \subseteq \mathcal{P}$. Assume that $Q \times 0 \subseteq \mathcal{P}$. It follows from 3.2 that $\mathcal{P} = Q_1 \times P_2$ for some ideal P_2 of Q_2. It is easily checked that P_2 must be prime. The case where $0 \times Q_2 \subseteq \mathcal{P}$ is similar. □

Similarly we prove the following theorem.

Theorem 4.3. Let Q_1 and Q_2 be quantales. Then an ideal $\mathcal{P}$ of $Q_1 \times Q_2$ is semi-prime if and only if $\mathcal{P}$ has the form $P_1 \times Q_2$ where P_1 is a semi-prime ideal of Q_1 or $Q_1 \times P_2$ where P_2 is a semi-prime ideal of Q_2.

Lemma 4.4. For a commutative quantale Q_1, the following conditions are equivalent.

1. For each commutative quantale P, every maximal ideal of $Q_1 \times P$ has the form $M \times P$ or $Q_1 \times N$, where M (respectively N) is a maximal ideal of Q_1 (respectively P).
2. Every maximal ideal of $Q_1 \times Q_1$ has the form $M \times Q_1$ or $Q_1 \times M$ where M is a maximal ideal of Q_1.
3. Every maximal ideal of $Q_1 \times Q_1$ is a sub-product.
4. For each commutative quantale P, every maximal ideal of $Q_1 \times P$ is a sub-product.

Proof. The implication $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are clear.
(4) $\Rightarrow$ (1). Let P a commutative quantale and let $\mathcal{M}$ be a maximal ideal of $Q_1 \times P$; (4) implies that $\mathcal{M}$ is a sub-product, then $\mathcal{M} = M \times N$, where M is an ideal of Q_1 and an N is an ideal of P. We have the inclusion $M \times N \subseteq M \times P$ and the inclusion $M \times N \subseteq Q_1 \times N$, so $\mathcal{M}$ has the form $M \times P$, where M is a maximal ideal of Q_1 or $\mathcal{M}$ has the form $Q_1 \times N$, where N is a maximal ideal of P. □

Proposition 4.5. For a commutative unital quantale Q_1, the following conditions are equivalent.

1. Every maximal ideal of Q_1 is prime.
2. For each commutative quantale Q_2, every maximal ideal of $Q_1 \times Q_2$ is prime.
3. Every maximal ideal of $Q_1 \times Q_1$ is prime.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{M}$ a maximal ideal of $Q_1 \times Q_2$; used 3.2 and conclude that $\mathcal{M}$ has the form $M \times Q_2$ where M is a maximal ideal of Q_1; so M maximal and used (1) and conclude that M is prime.
(2) $\Rightarrow$ (3) is clear.
(3 $\Rightarrow$ (1). Let M be a maximal ideal of Q_1; then 3.3 implies that $M \times Q_1$ is a maximal ideal of $Q_1 \times Q_1$; (3) implies that $M \times Q_2$ is a prime ideal of $Q_1 \times Q_1$. Lemma 3.3 implies that M is prime. □

Definition 4.6. For any commutative unitary quantale Q an I an ideal of Q, the radical of I is the the denoted by $\mathrm{Rad}(I)$ and defined by $\mathrm{Rad}(I) =: \{a \in Q : a^n \in I \text{ for some } n \in \mathbb{N}\}$. Note that for an element a of Q, $a^n = a \circ a \cdots \circ a$. An ideal I of Q is said to be radical if $\mathrm{Rad}(I) = I$.

Proposition 4.7. Let Q be a unitary commutative quantale and I be an ideal of Q; the following results are hold.
(1) The subset $\mathrm{Rad}(I)$ is an ideal of Q contain I.
(2) If I is a prime ideal, then I is radical.
(3) The correspondence $\mathrm{Rad}(-)$ defined on the set of all ideals Q is an idempotent endofunctor.
(4) The correspondence $\mathrm{Rad}(-)$ is a closure operator.
(5) If I is a primary ideal, then the radical of I is a prime ideal of Q.
(6) The radical of I is the intersection of all prime ideals of Q containing I.

Proof. (1) It is easy with $n = 1$ to see that I is a subset of $\mathrm{Rad}(I)$. Let x and y be elements of $\mathrm{Rad}(I)$, q be an element of Q. There exists the positive integer n and m such that x^n and y^m are elements of I. We have $(q \circ x)^n = q^n \circ x^n$ is an element of I, so $q \circ x$ is an element of $\mathrm{Rad}(I)$. If q is such that $q \leq x$, so $q^n \leq x^n$ an conclude that q is an element of $\mathrm{Rad}(I)$. To finish, see that x^{n+m} and y^{n+m} are elements of I and $x^{n+m} \vee y^{n+m}$ is an element of I. We remark that $(x \vee y)^{n+m} = x^{n+m} \vee y^{n+m} \vee (x \circ y)^{n+m}$ is an element of I and conclude that $x \vee y$ is an element of $\mathrm{Rad}(I)$.
(2) Assume that I is a prime ideal. Let x be an element of $\mathrm{Rad}(I)$ and n be a positive integer number such that x^n is an element of I. We have $x^n = x \circ x^{n-1}$ is an element of I. By induction on n, we prove that x is an element of I because I is prime.
(3) Let J be an ideal of Q such that $I \subseteq J$; let x be an element of $\mathrm{Rad}(I)$ and n be a positive integer number such that x^n is an element of I, so x^n is an element of J and conclude that x is an element of $\mathrm{Rad}(J)$. Now, let a be an element of $\mathrm{Rad}(\mathrm{Rad}(I))$; there is a positive integer number m such that a^m is an element of $\mathrm{Rad}(I)$; there is a positive integer number n such $(a^m)^n$ is in I; $(a^m)^n = a^{nm}$ is in I implies that a is an element of $\mathrm{Rad}(I)$, so $\mathrm{Rad}(\mathrm{Rad}(I))$=$\mathrm{Rad}(I)$.
(4) Follows by (1) and (3).

(5) Let a and b be two elements of Q such that $a \circ b$ is an element of Rad(I); so there is a positive number n such that $(a \circ b)^n = a^n \circ b^n$ is an element of I, so there is a positive integer m such that a^{nm} is an element of I or b^{nm} is an element of I. We say that a or b is an element of Rad(I).
(6) Let P be a prime ideal of Q containing I; If x is an element of Rad(I) and n be an entire number such that x^n is an element of I, the element x^n is in P and conclude that x is an element of P. □

It is not hard to show the following proposition.

Proposition 4.8. For each ideals I and J of a commutative quantale Q, Rad$(I \cap J)$ =Rad$(I)\cap$ Rad(J).

Theorem 4.9. Let P and Q be commutative quantales. The radical ideals of $P \times Q$ have the form $I \times J$ where I is a radical ideal of P and J is a radical ideal of Q.

Proof. Let K be a radical ideal of $P \times Q$. We may assume that K is a sub-product of the form $I \times J$ where I is an ideal of P and J an ideal of Q. So I is an intersection of prime ideals, each of which is a sub-product. So $K = I \times J$ is a sub-product where I and J is either the whole quantale or an intersection of prime ideals. In either case I and J is a radical ideal. □

Our next goal is to characterize the commutative quantales Q with the property that for each commutative quantale P, every primary ideal of is a sub-product. We need the following lemma.

Lemma 4.10. Let P and Q be commutative quantales.
(1) If A is an ideal with $A \neq Q$ and Rad$(A) = Q$, then A is primary.
(2) If K is a primary ideal of $P \times Q$ with Rad$(K) \neq P \times Q$, then either $K = I_1 \times Q$ where I_1 is a primary ideal of P or $K = P \times I_2$ where I_2 is a primary ideal of Q.

Proof. (1) Assume that a and b are elements of Q such that $a \circ b$ is an element of A. Then Rad$(A) = Q$ gives b^n is an element of A for some $n \geq 1$ regardless of whether a is an element of A or not. (2) Now Rad(K) is a prime ideal of $P \times Q$, so either Rad$(K) = A \times Q$ where A is a prime ideal of P or Rad$(K) = P \times B$ where B is a prime ideal of Q. Without loss of generality we may assume that Rad$(K) = A \times Q$ where A is a prime ideal of P. Let x be an element of $P - A$; so $(x, \perp)$ is not an element of Rad(A). Let q be an element of Q. Then $(\perp, q) \circ (x, \perp) = (\perp, \perp)$ is an element of A and $(x, \perp)$ is not element of Rad(A), so $(\perp, q)$ is an element of A, since A is primary. Hence $0 \times Q$ is a subset of A. So by Proposition 1, $Q = A \times Q$ for some ideal A of P which is easily seen to be primary. □

5 Conclusion

In view of the results we obtained in this paper and those obtained by other authors on quantales, we see a lot of similarities between the category of quantales and the category of rings. We have established that when we give ourselves two unital quantales Q_1 and Q_2, the ideals of the direct product $Q_1 \times Q_2$ are sub-products, that is to say of the form $K_1 \times K_2$ where K_i is ideal of Q_i for $i \in \{1, 2\}$. We have also established that the prime ideals (resp. semi-prime ideal, resp maximal ideals) of $Q_1 \times Q_2$ are of the form $P_1 \times Q_2$ or $Q_1 \times P_2$ where P_i is a prime ideal (resp. semi prime ideal, resp. maximal ideal) of Q_i for $i = 1, 2$.

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