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On the connections between Fibonacci and Mulatu Numbers

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Abstract: In this work, we present a detailed study of the Fibonacci–Mulatu sequence, $\{FM_n\}_{n\geq 0}$, defined recursively by $FM_{n+2} = FM_{n+1} + FM_n$ with initial terms $FM_0 = 4$ and $FM_1 = 1$. We establish several properties of this sequence, elucidating the connections it evinces with the classical Fibonacci and Fibonacci–Lucas sequences. Additionally, we derive an expression for negative indexes. Specifically, we derive the Binet formula for this sequence and determine the limit for quotients involving both positive and negative indexes. Additionally, we construct three types of generating functions–ordinary, exponential, and Poisson–and examine classical identities, including those of d'Ocagne, Catalan, Cassini, Melham, Cesàro, as well as the convolution identity. We further calculate partial sums and alternating partial sums of this sequence.

Keywords: Generating function; Fibonacci numbers; Fibonacci-type numbers; Golden ratio.

Classification MSC: 11B39; 11B37.

1 Introduction and Background

According to the literature and history of mathematics, the Fibonacci sequence was initially delineated in Liber Abaci (1202) by Leonardo of Pisa, an Italian mathematician who is also known as Fibonacci. He used this sequence to model the growth of a rabbit population under the most optimal conditions. Each number in the sequence is the sum of the two preceding numbers, so the Fibonacci sequence, denoted as $\{F_n\}_{n\geq 0}$, is defined by the recurrence relation $F_{n+2} = F_n + F_{n+1}$ for $n \geq 0$, with initial values $F_0 = 0$ and $F_1 = 1$. See [1–8] and [9] among others in order to learn more about this sequence. The Fibonacci sequence is listed as $A000045$ in the OEIS [10]. The first eight terms are: $0, 1, 1, 2, 3, 5, 8, 11, \ldots$.

Edouard Anatole Lucas (1842–1891) was a prominent mathematician who made significant contributions to the study of the Fibonacci sequence. In his work, he established a special type of Fibonacci sequence, denoted by $\{L_n\}_{n\geq 0}$. The universal rule of the Fibonacci sequence is as follows $L_{n+2} = L_n + L_{n+1}$ for $n \geq 0$, with initial terms $L_0 = 2$ and $L_1 = 1$. See [1, 2, 4] and [7], among others, to gain a deeper understanding of this sequence. The Lucas sequence begins as $2, 1, 3, 4, 7, 11, 18, \ldots$ and is known as the Fibonacci–Lucas sequence, listed as $A000032$ in the OEIS [10].

There are many different generalizations of the Fibonacci and the Fibonacci–Lucas numbers. Many authors have studied some generalizations of the Fibonacci sequence, either by preserving

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the original recurrence relation while modifying the initial terms or by maintaining the initial terms while introducing slight modifications to the recursive relation.

So, by modifying the initial conditions of the Fibonacci sequence, we can generate new sequences. Recently, the Professor Mulatu Lemma of Savannah State University introduced the Mulatu numbers, denoted by FM_n (see, for instance, the works of [11, 12], and [13]). This sequence follows the same Fibonacci recurrence relation $FM_{n+2} = FM_n + FM_{n+1}$ for $n \geq 0$, with initial values $FM_0 = 4$ and $FM_1 = 1$. This sequence starts with the terms $4, 1, 5, 6, 11, 17, 35, \ldots$ This sequence is cataloged as $A022095$ in the OEIS [10]. For the purpose of this discussion, this will be referred to as the Fibonacci–Mulatu numbers. So, the Mulatu sequence is a class of sequences related to the Fibonacci sequence, just as the Fibonacci–Lucas sequence.

In this context, we simplify the notation by denoting the sequence $\{F_n^*\}_{n\geq 0}$ as $\{F_n\}$, $\{L_n\}$ or $\{FM_n\}$, depending on the specific case. Thus, the sequence $\{F_n^*\}_{n\geq 0}$ is defined by the recurrence relation:

$$F_{n+2}^* = F_n^* + F_{n+1}^* \quad \text{for } n \geq 0 \ , \tag{1.1}$$

with initial conditions F_0^* and F_1^*. When the initial terms are set to $F_0^* = F_0 = 0$ and $F_1^* = F_1 = 1$, the sequence corresponds to the well-known Fibonacci numbers. On the other hand, when the initial values are $F_0^* = L_0 = 2$ and $F_1^* = L_1 = 1$, the sequence represents the Fibonacci–Lucas numbers. Finally, when the initial elements are $F_0^* = FM_0 = 4$ and $F_1^* = FM_1 = 1$, the sequence represents the Fibonacci–Mulatu numbers. Thus, depending on the choice of initial terms, $\{F_n^*\}_{n\geq 0}$ can either generate the classic Fibonacci sequence, or the Fibonacci–Lucas, or the Fibonacci–Mulatu sequence, each following the same recurrence relation but starting from different initial values.

Despite differing initial conditions, both $\{L_n\}_{n\geq 0}$ and $\{FM_n\}_{n\geq 0}$ are Fibonacci-like sequence, because this sequence follows the same recurrence as the Fibonacci sequence. Both the Fibonacci and Fibonacci–Lucas numbers are known to satisfy numerous mathematical identities, many of which have been discovered over the centuries. Like the Fibonacci and Fibonacci–Lucas sequences, the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ exhibits notable properties and patterns that have generated significant interest within the mathematical community (see [14–17, 19] and [20]).

Recent studies have made important advances in understanding the connections between two linear recurrence sequences, examining various aspects of their properties. These investigations have particularly focused on sequences in which terms are expressed as sums of terms from other sequences. For a more detailed discussion of recent work related to this problem, see references [21–23] and [24]. The main objective of this article is to categorize the Mulatu sequence by the linear combination of two Fibonacci-type sequences: the classical and the Lucas. This paper represents a contribution to the domain of number theory, prompting further investigation of the interrelationships between the three sequences.

Our paper is organized as follows: in the following Section, we present the Binet formula and derive an expression for negative indexes in terms of Fibonacci and Fibonacci–Lucas numbers. In Section 3, we explore the definition of the generating function and obtain the ordinary, exponential, and Poisson generating functions. Furthermore, we establish an expression involving hyperbolic functions. In Section 4, we establish some relations for the Fibonacci–Mulatu sequence, which are analogous to certain known identities of the Fibonacci sequence. In Section 5, we present some classic identities concerning the Fibonacci–Mulatu sequence, including the Tagiuri–Vajda identity, the Catalan identity, the Cassini identity and the d'Ocagne identity. Finally, in Section 6, we present results on partial sums of the Fibonacci–Mulatu sequence and partial sums with alternating terms.

2 The Binet formula and Mulatu with negative subscript

In this section, we present some known results connecting Fibonacci and Fibonacci–Lucas numbers as auxiliary results to be used in the following sections. We also provide a discussion on Binet–like formulas and derive the Binet formula for the Fibonacci–Mulatu sequence.

2.1 Auxiliary results

This article examines the relationships among Fibonacci numbers, Fibonacci–Lucas numbers, and Fibonacci–Mulatu numbers by considering the following identities for non-negative integers m and n:

[4, Equation 5.15] $F_{n-1} + F_{n+1} = L_n$; (2.1)

[4, Equation 5.18] $L_{n-1} + L_{n+1} = 5F_n$; (2.2)

[8, Equation 1.8] $F_{m-1}F_n + F_mF_{n+1} = F_{m+n}$; (2.3)

[4, Equation 5.19] $F_{-n} = (-1)^{n+1}F_n$; (2.4)

[4, Equation 5.20] $L_{-n} = (-1)^n L_n$ (2.5)

[4, Equation 5.16] $F_{n+4} = 3F_{n+2} - F_n$, (2.6)

[15, Theorem 2] $FM_n = F_n + 4F_{n-1}$; (2.7)

where $\{F_n\}_{n\geq 0}$ is the classical Fibonacci sequence, and $\{L_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence, and $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

The next result will be useful to us later, and we get it directly from the definition of the Fibonacci–Lucas sequence.

Lemma 2.1. *For all non-negative integers n, the following identity holds:*

$$L_{n+4} = 3L_{n+2} - L_n \ ; \tag{2.8}$$

where $\{L_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence.

For further details on the Fibonacci and Fibonacci–Lucas sequences, see [4, 7–9, 25], and [26] among others.

The next auxiliary result establishes identities involving the Fibonacci–Mulatu numbers with products of several other sequences.

Lemma 2.2. *[15, Theorem 5] For non-negative integer values of n and for any integer k, and for the $\{FM_n\}_{n\geq 0}$ Fibonacci–Mulatu sequence, we have*

$$FM_{n+k} = F_{k-1}FM_n + F_kFM_{n+1}, \tag{2.9}$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Remark 2.3. The Lemma 2.2 is a particular case of the general result due to Vajda [7, Equation (8)]. Namely, for any Fibonacci-type sequence $\{G_n\}_{n\geq 0}$ holds

$$G_{n+m} = F_{m-1}G_n + F_mG_{n+1},$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Now, the auxiliary result:

Lemma 2.4. *[7, Equation 18] For any Fibonacci-type sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$, the following identity holds:*

$$G_{n+h}H_{n+k} - G_nH_{n+h+k} = (-1)^n(G_hH_k - G_0H_{h+k}). \tag{2.10}$$

The Partial sum of Fibonacci–Mulatu numbers, the partial sum of the Fibonacci–Mulatu numbers with odd indexes, and the partial sum of the Fibonacci–Mulatu numbers with even indexes were respectively determined in [19]: Theorem 1, Theorem 2 and Theorem 3. Namely

Lemma 2.5. *[19] Let $\{FM_n\}_{n\geq 0}$ be the Fibonacci–Mulatu sequence. For all non-negative n integers, we have the following formulas:*

a) $\displaystyle\sum_{k=0}^{n} FM_k = FM_{n+2} - 1,$

b) $\displaystyle\sum_{k=0}^{n} FM_{2k} = FM_{2n+1} + 3,$

c) $\displaystyle\sum_{k=0}^{n} FM_{2k+1} = FM_{2n+2} - 4.$

Remark 2.6. The item (a) in Lemma 2.5 is a specific case of the general result due to [4, Exercises 25.15]. Item (b) can be obtained using a procedure similar to that used in item (a), and item (c) is obtained by combining items (a) and (b).

2.2 The Binet formula

The Binet formula is presented as a simple and efficient method for calculating the n-th term of a sequence (see in the mathematical literature the works of [27, 28] and [29], among others). Rather than an iterative process of calculating each previous term, this formula allows the desired term to be determined directly, thus bypassing the need to iterate through the sequence. Consequently, the n-th term of a sequence can be calculated more quickly and easily by using this expression in closed form.

The Equation (1.1) has a corresponding Fibonacci–Mulatu characteristic equation $x^2-x-1=0$, which has distinct real roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. According to standard results on linear recurrence relations, if the Horadam characteristic equation $x^2+px+q=0$ has distinct roots α and β, then the general solution to the recurrence relation is given by:

$$x_n = C_1(\alpha)^n + C_2(\beta)^n,$$

where C_1 and C_2 are constants to be determined, and α and β are the roots of the characteristic equation. To find the constants C_1 and C_2, we use the initial conditions F_0^* and F_1^*, we can set up the following system of linear equations:

$$\begin{cases} F_0^* = C_1(\alpha)^0 + C_2(\beta)^0 = C_1 + C_2\ , \\ F_1^* = C_1(\alpha)^1 + C_2(\beta)^1 = C_1\alpha + C_2\beta\ . \end{cases}$$

Solving this system will provide the values of C_1 and C_2, which completely determine the sequence $\{F_n^*\}$ based on the given recurrence relation.

Fibonacci numbers F_n have two initial terms, $F_0 = 0$ and $F_1 = 1$, so

$$[7, \text{Equation } 58] \quad F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \; ; \tag{2.11}$$

Fibonacci–Lucas numbers L_n have two initial terms, $L_0 = 2$ and $L_1 = 1$, so

$$[7, \text{Equation } 59] \quad L_n = \alpha^n + \beta^n \; ; \tag{2.12}$$

Fibonacci–Mulatu numbers FM_n have two initial terms, $FM_0 = 4$ and $FM_1 = 1$, so

$$[21, \text{Equation } 15] \quad FM_n = \frac{10-\sqrt{5}}{5}\alpha^n + \frac{10+\sqrt{5}}{5}\beta^n \; . \tag{2.13}$$

The results expressed in Equations (2.11), (2.12) and (2.13) are derived from the result presented in [4, Theorem 7.4].

The next statement gives the Binet formula for the sequence $\{F_n^*\}_{n\geq 0}$, and can be found in [30, Lemma 1], which is also derived from [4, Theorem 7.4].

Lemma 2.7 (Binet-like formula)**.** *Let $c = F_1^* - F_0^*\beta$ and $d = F_1^* - F_0^*\alpha$. Then*

$$F_n^* = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}, \tag{2.14}$$

where α is the golden ratio $(1+\sqrt{5})/2$, β is its conjugate $(1-\sqrt{5})/2$, and $\{F_n^*\}_{n\geq 0}$ the Fibonacci-type sequence.

Note that $c = 1 - 0\beta = 1$ and $d = 1 - 0\alpha = 1$ if $\{F_n^*\}$ is the $\{F_n\}$ Fibonacci numbers, and we give $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Already $c = 1 - 2\beta$ and $d = 1 - 2\alpha$ if $\{F_n^*\}$ is the $\{L_n\}$ Fibonacci–Lucas numbers; while $c = 1 - 4\beta$ and $d = 1 - 4\alpha$ if $\{F_n^*\}$ is the $\{FM_n\}$ Fibonacci–Mulatu numbers. And more,

$$cd \;=\; [F_1^* - F_0^*\beta][F_1^* - F_0^*\alpha] = (F_1^*)^2 - F_0^*F_1^*(\alpha+\beta) + (F_0^*)^2\alpha\beta = (F_1^*)^2 - F_0^*F_1^* - (F_0^*)^2 \; ,$$

since $\alpha\beta = -1$ and $\alpha + \beta = 1$.

According to [4], this constant $\mu = cd$ occurs in many of the formulas for Fibonacci-type numbers and is referred to as the characteristic of the Fibonacci-type sequence. For instance, the characteristic of the Fibonacci sequence is $\mu = 1$, that of the Fibonacci–Lucas sequence is $\mu = -5$, and that of the Fibonacci–Mulatu sequence is $\mu = -19$.

The next claim expresses an alternative closed form for the Fibonacci–Mulatu numbers $\{FM_n\}_{n\geq 0}$, in order to calculate the n-th term of this sequence.

Theorem 2.8. *Let be $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ the roots of the quadratic equation $x^2 - x - 1 = 0$. Then, for all $n \geq 0$, we have*

$$FM_n = 2(\alpha^n + \beta^n) - \frac{\alpha^n - \beta^n}{\alpha - \beta} \; , \tag{2.15}$$

where $\{FM_n\}_{n\geq 0}$ *is the Fibonacci–Mulatu sequence.*

Proof. Let

$$U_n = 2(\alpha^n + \beta^n) - \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $n \geq 0$. Then

$U_0 = 2(\alpha^0 + \beta^0) - \dfrac{\alpha^0 - \beta^0}{\alpha - \beta} = 2(1+1) = 4$ and $U_1 = 2(\alpha + \beta) - \dfrac{\alpha - \beta}{\alpha - \beta} = 2(1) - 1 = 1$, where we use the fact $\alpha + \beta = 1$. Now, suppose $n \geq 2$. Then

$$\begin{aligned} U_{n-1} + U_{n-2} &= 2(\alpha^{n-1} + \beta^{n-1}) - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + 2(\alpha^{n-2} + \beta^{n-2}) - \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} \\ &= 2(\alpha^{n-2}(\alpha+1) + \beta^{n-2}(\beta+1)) - \frac{\alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)}{\alpha - \beta} \\ &= 2(\alpha^{n-2}\alpha^2 + \beta^{n-2}\beta^2) - \frac{\alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2}{\alpha - \beta} \\ &= 2(\alpha^n + \beta^n) - \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n\ . \end{aligned}$$

We use the hypothesis that both α and β are solutions of the equation $x^2 - x - 1 = 0$, which implies that $1 + \alpha = \alpha^2$ and $1 + \beta = \beta^2$. Thus, U_n satisfies the Fibonacci–Mulatu recurrence with the same two initial conditions. Therefore, we conclude that $U_n = FM_n$. ☐

As a consequence of Theorem 2.8, we have the following result.

Corollary 2.9. *[17] The Fibonacci–Lucas sequence is the arithmetic mean of the Fibonacci and Fibonacci–Mulatu sequences, that is,*

$$L_n = \frac{F_n + FM_n}{2}, \quad n \geq 0\ , \tag{2.16}$$

where, respectively, $\{F_n\}$, $\{L_n\}$ *and* $\{FM_n\}$ *are Fibonacci, Fibonacci–Lucas and Fibonacci–Mulatu sequences.*

Remark 2.10. According [18], we will say that the Corollary 2.9 establishes the condition for a number sequence $C = \{c_n\}_{n\geq 0}$ to be of type A-Mulatu, that is, given the sequences $A = \{a_n\}_{n\geq 0}$ and $B = \{b_n\}_{n\geq 0}$ with the same recurrence relation, with B being of type A-Lucas, if $2b_n = a_n + c_n$, for all n, then the sequence C is A-Mulatu.

2.3 Fibonacci–Mulatu with negative subscript

We have seen that the Fibonacci and Fibonacci–Lucas sequences can be extended to indexes negative using the identities $F_{-n} = (-1)^{n+1}F_n$, Equation (2.4), for the Fibonacci numbers and $L_{-n} = (-1)^n L_n$, Equation (2.5), for the Fibonacci–Lucas numbers, where $n \geq 0$.

We want to give meaning to the sequence FM_n for every integer number n, and for the recurrence to remain valid. The Fibonacci–Mulatu sequence is characterized by a recurrence relation, which can be expressed as follows $FM_n = FM_{n-1} + FM_{n-2}$ with initial conditions $FM_0 = 4$ and $FM_1 = 1$. To extend the sequence to indexes negative, we use the modified recurrence relation:

$$FM_{n-2} = FM_n - FM_{n-1}.$$

Using this, we compute the Fibonacci–Mulatu terms for the indexes negative. The calculations are as follows:

$$\begin{array}{lclclclcl}
FM_{-1} & = & FM_1 - FM_0 & = & 1-4 & = & -3 & = & -(1+2\cdot 1)\ , \\
FM_{-2} & = & FM_0 - FM_{-1} & = & 4-(-3) & = & 7 & = & (5+2\cdot 1)\ , \\
FM_{-3} & = & FM_{-1} - FM_{-2} & = & -3-7 & = & -10 & = & -(6+2\cdot 2)\ , \\
FM_{-4} & = & FM_{-2} - FM_{-3} & = & 7-(-10) & = & 17 & = & (11+2\cdot 3)\ , \\
FM_{-5} & = & FM_{-3} - FM_{-4} & = & -10-17 & = & -27 & = & -(17+2\cdot 5)\ ,
\end{array}$$

and so on.

The Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ can be extended to negative subscripts by the following result.

Proposition 2.11. *Let $n \geq 1$, then the negative index n-th numbers are defined as*

$$FM_{-n} = (-1)^n (FM_n + 2F_n)\,, \tag{2.17}$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence, and $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

Proof. By Corollary 2.9 we have $2L_n - F_n = FM_n$. So, take into account Equations (2.4) and (2.5)

$$\begin{array}{rcl}
FM_{-n} & = & 2L_{-n} - F_{-n} \\
 & = & 2(-1)^n L_n - (-1)^{n+1} F_n \\
 & = & (-1)^n (2L_n + F_n)\ ,
\end{array}$$

as $FM_n = F_n - 4F_{n-1}$, Equation (2.7), we get

$$\begin{array}{rcl}
FM_{-n} & = & (-1)^n (2L_n + FM_n - 4F_{n-1}) \\
 & = & (-1)^n [FM_n + 2(L_n - 2F_{n-1})] = (-1)^n [FM_n + 2(F_{n-1} + F_{n+1} - 2F_{n-1})] \\
 & = & (-1)^n [FM_n + 2(F_{n+1} - F_{n-1})] = (-1)^n [FM_n + 2F_n]\ ,
\end{array}$$

in which we used the Equation (2.1), and we get the result. ☐

It follows from the previous result that the Fibonacci–Mulatu sequence with a negative index also has the following property:

Proposition 2.12. *For negative indexes, the Fibonacci–Mulatu sequence satisfies the following identity:*

$$FM_{-n} = (-1)^n (2L_n + F_n),$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence, $\{L_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence, and $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

Proof. By Corollary 2.9, we have $2L_n + F_n = 2L_n - F_n + 2F_n = FM_n + 2F_n$, and the result follows by Proposition 2.11. $\square$

3 The generating function

In this section, we will present both the exponential generating function and the generating function for the Fibonacci–Mulatu sequence. In the literature, the function $G_{a_n}(x)$ is known as the generating function for the sequence $\{a_n\}_{n\geq 0}$ and is defined by

$$G_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots .$$

Additionally, our next result provides the generating function for the Fibonacci–Mulatu sequence.

Proposition 3.1. *The generating function for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ denoted by $G_{FM_n}(x)$, is given by*

$$G_{FM_n}(x) = \frac{4-3x}{1-x-x^2} . \tag{3.1}$$

Proof. Indeed,

$$\begin{aligned} G_{FM_n}(x) &= FM_0 + FM_1 x + FM_2 x^2 + FM_3 x^3 + \cdots + FM_n x^n + \cdots \\ xG_{FM_n}(x) &= FM_0 x + FM_1 x^2 + FM_2 x^3 + FM_3 x^4 + \cdots + FM_n x^{n+1} + \cdots \\ x^2 G_{FM_n}(x) &= FM_0 x^2 + FM_1 x^3 + FM_2 x^4 + FM_3 x^5 + \cdots + FM_n x^{n+2} + \cdots . \end{aligned}$$

Then,

$$\begin{aligned} G_{FM_n}(x) - xG_{FM_n}(x) - x^2 G_{FM_n}(x) &= FM_0 + (FM_1 - FM_0)x + (FM_2 - FM_1 - FM_0)x^2 + \\ &+ (FM_3 - FM_2 - FM_1)x^3 + (FM_4 - FM_3 - FM_2)x^4 + (FM_5 - FM_4 - FM_3)x^5 + \cdots \\ &= FM_0 + (FM_1 - FM_0)x \\ &= 4 + (1-4)x = 4 - 3x . \end{aligned}$$

Since, $FM_0 = 4$ and $FM_1 = 1$, we have $(1 - x - x^2)G_{FM_n}(x) = 4 - 3x$, or

$$G_{FM_n}(x) = \frac{4-3x}{1-x-x^2} ,$$

with $1 - x - x^2 \neq 0$, which concludes the proof. $\square$

In the literature, the exponential generating function $E_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is defined as the power series

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} .$$

The Fibonacci numbers F_n and Fibonacci–Lucas numbers L_n can also be defined using their exponential generating functions [4, Section 13.8],

$$E_{F_n}(x) = \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} \ ; \tag{3.2}$$

$$E_{L_n}(x) = \sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} \ . \tag{3.3}$$

By combining Equations (2.15), (2.16), (3.2) and (3.3), we obtain the following result.

Proposition 3.2. *For all $n \geq 0$ the exponential generating function for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ is*

$$E_{FM_n}(x) = \sum_{n=0}^{\infty} FM_n \frac{x^n}{n!} = \frac{(2\sqrt{5}-1)e^{\alpha x} + (2\sqrt{5}+1)e^{\beta x}}{\alpha - \beta},$$

where α is the golden ratio $(1+\sqrt{5})/2$ and β is its conjugate $(1-\sqrt{5})/2$.

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is defined as $P_{a_n}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} e^{-x}$. This function represents the sequence $\{a_n\}_{n\geq 0}$ in terms of the parameter x. There is a notable connection between the exponential generating function $E_{a_n}(x)$ and the Poisson generating function $P_{a_n}(x)$, which can be expressed by the following equation $P_{a_n}(x) = e^{-x} E_{a_n}(x)$.

As a result, the corresponding Poisson generating function is derived.

Corollary 3.3. *For all $n \geq 0$ the Poisson generating function for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ is*

$$P_{FM_n}(x) = \frac{(2\sqrt{5}-1)e^{-\beta x} + (2\sqrt{5}+1)e^{-\alpha x}}{\alpha - \beta},$$

where α is the golden ratio $(1+\sqrt{5})/2$ and β is its conjugate $(1-\sqrt{5})/2$.

Proof. Indeed, it follows from Proposition 3.2 and the fact $\alpha + \beta = 1$. ☐

As we can see in [4], the sequences F_n and L_n are related to the hyperbolic functions sinh and cosh. This follows from the Proposition 3.2 that the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ satisfies the following result.

Corollary 3.4. *The Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ satisfies the following identity:* $\sum_{n=0}^{\infty} 2^n FM_n \frac{x^n}{n!} = 2e^x \left(2\cosh(\sqrt{5}x) - \frac{1}{\sqrt{5}} \sinh(\sqrt{5}x) \right)$.

Proof. It follows from the Proposition 3.2 that $\sum_{n=0}^{\infty} \frac{FM_n}{n!} x^n = 2(e^{\alpha x} + e^{\beta x}) - \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$. We can see that

$$2(e^{\alpha x} + e^{\beta x}) = 4e^{\frac{x}{2}} \cosh\left(\frac{\sqrt{5}}{2}x\right) \quad \text{and} \quad \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \frac{2}{\sqrt{5}} e^{\frac{x}{2}} \sinh\left(\frac{\sqrt{5}}{2}x\right).$$

By substituting x with $2x$, we obtain:

$$\sum_{n=0}^{\infty} 2^n FM_n \frac{x^n}{n!} = 2e^x \left(2\cosh(\sqrt{5}x) - \frac{1}{\sqrt{5}}\sinh(\sqrt{5}x)\right) ,$$

this completes the proof. □

4 Some properties for Fibonacci–Mulatu numbers

In this section, for some well-known identities of the Fibonacci sequence $\{F_n\}_{n\geq 0}$, we will establish analogous relations for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$.

First, we present relations between the Fibonacci, Fibonacci–Lucas, and Fibonacci–Mulatu sequences.

Proposition 4.1. *For all integers n , then the $\{FM_n\}_{n\geq 0}$ Fibonacci–Mulatu sequence satisfies the following identities:*

$$a)\ \ FM_{n-1} + FM_{n+1} = 10F_n - L_n \tag{4.1}$$

$$b)\ \ FM_{n-1} + FM_{n+1} = 9F_n - 2F_{n-1} \tag{4.2}$$

$$c)\ \ FM_{n-1} + FM_{n+1} = 19L_n - 10FM_n \tag{4.3}$$

where, respectively, $\{F_n\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$ are Fibonacci, Fibonacci–Lucas sequences.

Proof. It follows from Equations (2.1), (2.2) and (2.16):

$$\begin{aligned} FM_{n-1} + FM_{n+1} &= 2L_{n-1} - F_{n-1} + 2L_{n+1} - F_{n+1} = 2(L_{n-1} + L_{n+1}) - (F_{n-1} + F_{n+1}) \\ &= 2 \cdot 5F_n - L_n = 10F_n - L_n , \end{aligned}$$

and (4.1) follows.

By Equations (2.1) and (4.1), we have $10F_n - L_n = 10F_n - (F_{n+1} + F_{n-1}) = 10F_n - (F_n + F_{n-1} + F_{n-1}) = 9F_n - 2F_{n-1}$, that implies the equation (4.2).

As a consequence of Corollary 2.9, we obtain $10F_n - L_n = 10(2L_n - FM_n) - L_n = 20L_n - 10FM_n - L_n = 19L_n - 10FM_n$. which concludes the proof by the use of Equation (4.1). □

The next statement is a relation between the elements FM_n, FM_{n+2} and FM_{n+4}.

Proposition 4.2. *For all integers n , then the $\{FM_n\}_{n\geq 0}$ Fibonacci–Mulatu sequence satisfies the following identity:*
$FM_n = 3FM_{n+2} - FM_{n+4}$.

Proof. It follows from Equations (2.6), (2.8) and (2.16) that

$$\begin{aligned} FM_{n+4} &= 2L_{n+4} - F_{n+4} = 2(3L_{n+2} - L_n) - (3F_{n+2} - F_n) \\ &= 6L_{n+2} - 2L_n - 3F_{n+2} + F_n = 6L_{n+2} - 3F_{n+2} - 2L_n + F_n \\ &= 3(2L_{n+2} - F_{n+2}) - (2L_n - F_n) = 3FM_{n+2} - FM_n , \end{aligned}$$

which verifies the result. □

The next result shows the product of two consecutive elements of the Fibonacci–Mulatu sequence.

Proposition 4.3. *For all integers n, the Fibonacci–Mulatu sequence $\{FM_n\}_n$ satisfies the following identity*

$$FM_n \cdot FM_{n+1} = \frac{1}{5}\left[F_{2n+2} + 41F_{2n} + 19(-1)^n\right],$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. By Binet's formula we have

$$FM_nFM_{n+1} = \left[2(\alpha^n+\beta^n) - \frac{\alpha^n-\beta^n}{\alpha-\beta}\right]\left[2(\alpha^{n+1}+\beta^{n+1}) - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right] = I - J,$$

where

$$I = 2(\alpha^n+\beta^n)\left[2(\alpha^{n+1}+\beta^{n+1}) - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right]$$

and

$$J = \frac{\alpha^n-\beta^n}{\alpha-\beta}\left[2(\alpha^{n+1}+\beta^{n+1}) - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right].$$

Now,

$$\begin{aligned}
I &= 4(\alpha^{2n+1}+\beta^{2n+1}+(\alpha\beta)^n(\alpha+\beta)) - \frac{2[\alpha^{2n+1}-\beta^{2n+1}+(\alpha\beta)^n(\alpha-\beta)]}{\alpha-\beta}\\
&= 4(\alpha^{2n+1}+\beta^{2n+1}+(-1)^n) - \frac{2(\alpha^{2n+1}-\beta^{2n+1})}{\alpha-\beta} - 2(-1)^n\\
&= 4(\alpha^{2n+1}+\beta^{2n+1}) - \frac{2(\alpha^{2n+1}-\beta^{2n+1})}{\alpha-\beta} + 2(-1)^n,
\end{aligned}$$

and

$$\begin{aligned}
J &= 2(\alpha^{n+1}+\beta^{n+1})\frac{\alpha^n-\beta^n}{\alpha-\beta} - \frac{\alpha^n-\beta^n}{\alpha-\beta}\left[\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right]\\
&= \frac{2(\alpha^{2n+1}-\beta^{2n+1}+(\alpha\beta)^n(\beta-\alpha))}{\alpha-\beta} - \frac{\alpha^{2n+1}+\beta^{2n+1}-(\alpha\beta)^n(\alpha+\beta)}{(\alpha-\beta)^2}\\
&= 2\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta} - 2(-1)^n - \frac{\alpha^{2n+1}+\beta^{2n+1}}{(\alpha-\beta)^2} + \frac{(-1)^n}{(\alpha-\beta)^2}\\
&= 2\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta} - 2(-1)^n - \frac{\alpha^{2n+1}+\beta^{2n+1}}{5} + \frac{(-1)^n}{5}.
\end{aligned}$$

Then

$$\begin{aligned} I - J &= 4(\alpha^{2n+1} + \beta^{2n+1}) - \frac{4(\alpha^{2n+1} - \beta^{2n+1})}{\alpha - \beta} + \frac{\alpha^{2n+1} + \beta^{2n+1}}{5} + 4(-1)^n - \frac{(-1)^n}{5} \\ &= 4L_{2n+1} - 4F_{2n+1} + \frac{1}{5}L_{2n+1} + \frac{19}{5}(-1)^n \\ &= \frac{21}{5}L_{2n+1} - 4F_{2n+1} + \frac{19}{5}(-1)^n = \frac{1}{5}[21L_{2n+1} - 20F_{2n+1} + 19(-1)^n]\,, \end{aligned}$$

and the result immediately follows from the Equation (2.1). □

The next claim provides a formula for double addition for Mulatu numbers, and it follows from Lemma 2.2. This result will be employed to derive the Tagiuri-Vajda identity in the following section.

Proposition 4.4. *For non-negative integer values of n and m, and for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$, the following identity holds:*

$$2FM_{n+m} = L_m FM_n + 10F_m F_n - F_m L_n\,, \tag{4.4}$$

where $\{L_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence and $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Making $k = -m$ in Equation (2.9), and by using the Equations (2.4) and (2.1), we have $FM_{n-m} = (-1)^m(F_{m+1}FM_n - F_m FM_{n+1})$. Then $FM_{n+m} + (-1)^m FM_{n-m} = F_{m-1}FM_n + F_m FM_{n+1} + F_{m+1}FM_n - F_m FM_{n+1} = (F_{m-1} + F_{m+1})FM_n = L_m FM_n$ and

$$\begin{aligned} FM_{n+m} - (-1)^m FM_{n-m} &= F_{m-1}FM_n + F_m FM_{n+1} - F_{m+1}FM_n + F_m FM_{n+1} \\ &= (F_{m-1} - F_{m+1})FM_n + 2F_m FM_{n+1} = -F_m FM_n + 2F_m FM_{n+1} \\ &= F_m(FM_{n-1} - FM_{n+1}) + 2F_m FM_{n+1} = F_m FM_{n-1} - F_m FM_{n+1} + 2F_m FM_{n+1} \\ &= F_m FM_{n-1} + F_m FM_{n+1} = F_m(FM_{n-1} + FM_{n+1})\,. \end{aligned}$$

Now, summing the last two equations, we obtain $2FM_{n+m} = L_m FM_n + F_m(FM_{n-1} + FM_{n+1})$. By using the Equation (4.1), we obtain

$$2FM_{n+m} = L_m FM_n + F_m(FM_{n-1} + FM_{n+1}) = L_m FM_n + 10F_m F_n - F_m L_n,$$

as required. □

5 Some Classical Identities

In this section, we derive and discuss several classical identities related to the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$. These identities include well-known results such as the Tagiuri-Vajda identity, the Catalan identity, the Cassini identity, and the d'Ocganes identity. By exploring these identities, we aim to provide a deeper understanding of the properties and behaviors of the Fibonacci–Mulatu sequence.

First, the Tagiuri-Vajda's Identity for the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$.

Theorem 5.1. *Let r, s, k be any natural numbers. We have*

$$FM_{r+s}FM_{r+k} - FM_rFM_{r+s+k} = (-1)^{r+1}19F_sF_k \ , \tag{5.1}$$

where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence, and $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Taking $G_i = H_i = FM_i$ in Equation (2.10) we have
$FM_{n+h}FM_{n+k} - FM_nFM_{n+h+k} = (-1)^n(FM_hFM_k - FM_0FM_{h+k}) = (-1)^n(FM_hFM_k - 4FM_{h+k})$.

It follows from Equations (4.4) and (2.16) that

$$\begin{aligned}(-1)^n(FM_hFM_k - 4FM_{h+k}) &= (-1)^n(FM_hFM_k - 2L_kFM_h - 20F_kF_h + 2F_kL_h)\\ &= (-1)^n(FM_h(FM_k - 2L_k) - 2F_k(10F_h - L_h))\\ &= (-1)^n(-F_k(FM_h + 20F_h - 2L_h)) = (-1)^n(-F_k(FM_h - 2L_h + 20F_h))\\ &= (-1)^n(-F_k(-F_h + 20F_h)) = (-1)^n(-19F_kF_h) = (-1)^{n+1}19F_kF_h\end{aligned}$$

which completes the proof. ☐

Remark 5.2. The Tagiuri-Vajda identity for the Fibonacci and Fibonacci–Lucas sequences has already appeared in some previous work, for example [7, Equation 20a and 20b]. Let r, s, k be any natural numbers. We have the following identity:

$$\begin{aligned}F_{r+s}F_{r+k} - F_rF_{r+s+k} &= (-1)^rF_sF_k \ ; \qquad (5.2)\\ L_{r+s}L_{r+k} - L_rL_{r+s+k} &= (-1)^{r+1}5F_sF_k \ , \qquad (5.3)\end{aligned}$$

where $\{F_n\}$ is the Fibonacci sequence, and $\{L_n\}$ is the Fibonacci–Lucas sequence. We can combine the Equations (5.1), (5.2) and (5.3) into one in the following form: let r, s, k be any natural numbers. We have the following identity: $F^*_{r+s}F^*_{r+k} - F^*_rF^*_{r+s+k} = (-1)^{r+\xi}\mu F_sF_k$, where $\{F^*_n\}$ is the Fibonacci-type sequence, $\{F_n\}$ Fibonacci sequence, μ is the characteristic of Fibonacci-type sequence, and $\xi = 0$ if $\{F^*_n\}_{n\geq 0} = \{F_n\}_{n\geq 0}$ and $\xi = 1$ if $\{F^*_n\}_{n\geq 0} = \{L_n\}_{n\geq 0}$ or $\{F^*_n\}_{n\geq 0} = \{FM_n\}_{n\geq 0}$, as you can see in previous identities.

As a consequence of Tagiuri-Vajda's identity, Equation (5.1), the following results establish the d'Ocagne's identity, Catalan's identity, and Cassini's identity, respectively, for the Fibonacci–Mulatu sequence.

Proposition 5.3 (d'Ocagne's identity)**.** *Let m, n be non-negative integers, then*

$$FM_mFM_{n+1} - FM_nFM_{m+1} = (-1)^{n+1}19F_{m-n},$$

where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence, and $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Consider $r = m - n$ and $k = 1$ in Equation (5.1), then

$$FM_mFM_{n+1} - FM_nFM_{m+1} = (-1)^{n+1}19F_{m-n}F_1 = (-1)^{n+1}19F_{m-n},$$

which proves the result. ☐

Proposition 5.4 (Catalan's identity)**.** *Let n, r be non-negative integers, then*

$$FM_{n+k}FM_{n-k} - (FM_n)^2 = 19(-1)^{n+k}F_k^2 \ , \tag{5.4}$$

where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence, and $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Taking $r = -k$ in Equation (5.1), we have that

$$FM_{n-k}FM_{n+k} - FM_nFM_n = (-1)^{n+1}19F_{-k}F_k.$$

As $F_{-k} = (-1)^{k+1}F_k$, we have $FM_{n+k}FM_{n-k} - (FM_n)^2 = 19(-1)^{n+k+2}(F_k)^2 = 19(-1)^{n+k}(F_k)^2$ and the result follows. □

As a consequence of Catalan's identity, since $F_1 = 1$ and by making $k = 1$ in the equation (5.4), we have the following property.

Corollary 5.5 (Cassini-Simson's identity)**.** *For all non-negative integer n, we have*

$$(FM_n)^2 - FM_{n+1}FM_{n-1} = 19(-1)^n,$$

where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

Other consequence is the Cassini-Simson identity for subscripts even:

Corollary 5.6. *For all non-negative integers n, we have $(FM_{2n})^2 - FM_{2n+1}FM_{2n-1} = 19$, where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.*

Now, we present the Convolution identity for Fibonacci–Mulatu sequence.

Proposition 5.7 (Convolution's identity)**.** *Let m, n be non-negative integers, then the Fibonacci–Mulatu sequence $\{FM_n\}_{n\geq 0}$ satisfies the following identity*

$$FM_{m-1}FM_n + FM_mFM_{n+1} = FM_{m+n} + 4FM_{m+n-1} \, .$$

Proof. As $FM_n = F_n + 4F_{n-1}$, Equation (2.7) and (2.3). Then

$$\begin{aligned}
&FM_{m-1}FM_n + FM_mFM_{n+1} = (F_{m-1} + 4F_{m-2})(F_n + 4F_{n-1}) + (F_m + 4F_{m-1})(F_{n+1} + 4F_n)\\
&= F_{m-1}F_n + 4F_{m-1}F_{n-1} + 4F_{m-2}F_n + 16F_{m-2}F_{n-1} + F_mF_{n+1}\\
&\qquad\qquad + 4F_mF_n + 4F_{m-1}F_{n+1} + 16F_{m-1}F_n\\
&= F_{m-1}F_n + F_mF_{n+1} + 4(F_{m-1}F_{n-1} + F_mF_n)\\
&\qquad + 4(F_{m-2}F_n + F_{m-1}F_{n+1}) + 16(F_{m-2}F_{n-1} + F_{m-1}F_n)\\
&= F_{m+n} + 4F_{m+n-1} + 4(F_{m+n-1} + 4F_{m+n-2}) = FM_{n+m} + 4FM_{n+m-1},
\end{aligned}$$

as required. □

As a consequence of Proposition 5.7, the next result provides the convolution identity for the Fibonacci–Mulatu sequence in terms of the Fibonacci sequence.

Corollary 5.8. *The Fibonacci–Mulatu sequence* $\{FM_n\}_{n\geq 0}$ *satisfies the following:*

$$FM_{m-1}FM_n + FM_mFM_{n+1} = 9F_{m+n} + 8F_{m+n-2},$$

where $\{F_n\}_{n\geq 0}$ *is the Fibonacci sequence.*

Proof. It follows from Proposition 5.7 that

$$\begin{aligned} FM_{m-1}FM_n \quad &+ \quad FM_mFM_{n+1} = F_{m+n} + 8F_{m+n-1} + 16F_{m+n-2} \\ &= \quad F_{m+n} + (8F_{m+n-1} + 8F_{m+n-2}) + 8F_{m+n-2} = 9F_{m+n} + 8F_{m+n-2}\ , \end{aligned}$$

as required. ☐

A direct calculation yields the Melham identity.

Proposition 5.9 (Melham's identity)**.** *The Fibonacci–Mulatu sequence* $\{FM_n\}_{n\geq 0}$ *satisfies the following identity:*

$$FM_{n+1}FM_{n+2}FM_{n+6} - (FM_{n+3})^3 = 19(-1)^{n+1}FM_n.$$

Proof. Indeed we have

$$\begin{aligned} &FM_{n+1}FM_{n+2}(5FM_{n+2} + 3FM_{n+1}) - (FM_{n+2} + FM_{n+1})^3 \\ &\qquad = 2FM_{n+1}(FM_{n+2})^2 - (FM_{n+2})^3 - (FM_{n+1})^3 \\ &\qquad = (FM_{n+2})^2(2FM_{n+1} - FM_{n+2}) - (FM_{n+1})^3 \\ &\qquad = (FM_{n+1} + FM_n)^2(FM_{n+1} - FM_n) - (FM_{n+1})^3 \\ &\qquad = (FM_{n+1} + FM_n)((FM_{n+1})^2 - (FM_n)^2) - (FM_{n+1})^3 \\ &\qquad = FM_nFM_{n+1}(FM_{n+1} - FM_n) - (FM_n)^3 = FM_nFM_{n+1}FM_{n-1} - (FM_n)^3 \\ &\qquad = FM_n((FM_n)^2 - 19(-1)^n) - (FM_n)^3 = 19(-1)^{n+1}FM_n\ , \end{aligned}$$

where we apply Cassini-Simson's identity, Corollary 5.5, and the fact that $FM_{n+6} = 5FM_{n+2} + 3FM_{n+1}$. This result follows directly from the recurrence relation of FM_n. ☐

Proposition 5.10 (Gelin-Cesàro's identity)**.** *The Fibonacci–Mulatu sequence* $\{FM_n\}_{n\geq 0}$ *satisfies the following identity:*

$$FM_{n+2}FM_{n+1}FM_{n-1}FM_{n-2} - (FM_n)^4 = -19^2.$$

Proof. Using Catalan's identity (5.4) for $k = 2$ and $k = 1$ we obtain

$$\begin{aligned} FM_{n+2}FM_{n+1}FM_{n-1}FM_{n-2} - (FM_n)^4 &= FM_{n+2}FM_{n-2}FM_{n+1}FM_{n-1} - (FM_n)^4 \\ &= \left[(FM_n)^2 + 19(-1)^{n+4}(F_2)^2\right]\left[(FM_n)^2 + 19(-1)^{n+3}(F_1)^2\right] - (FM_n)^4 \\ &= (FM_n)^4 - 19^2(-1)^{2n} - (FM_n)^4 = -19^2\ , \end{aligned}$$

as required. ☐

Using the Binet formulas (2.15) once again, we derive another property of the Fibonacci–Mulatu sequences $\{FM_n\}_{n\in\mathbb{Z}}$, presented in the following proposition.

Proposition 5.11. *If FM_n are the n-th term of Fibonacci–Mulatu sequence, then*

$$\lim_{n\to\infty} \frac{FM_{n+1}}{FM_n} = \alpha \ , \tag{5.5}$$

and

$$\lim_{n\to\infty} \frac{FM_{-(n+1)}}{FM_{-n}} = -\alpha \ , \tag{5.6}$$

where α is the golden ratio $(1+\sqrt{5})/2$.

Proof. According to Binet's formula (2.15), we have that

$$\begin{aligned}\frac{FM_{n+1}}{FM_n} &= \frac{2(\alpha^{n+1}+\beta^{n+1}) - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}}{2(\alpha^n+\beta^n) - \frac{\alpha^n-\beta^n}{\alpha-\beta}} = \frac{\alpha^{n+1}\left[2+2(\frac{\beta}{\alpha})^{n+1} - \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}\right]}{\alpha^n\left[2+2(\frac{\beta}{\alpha})^{n} - \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}\right]} \\ &= \alpha\,\frac{2+2(\frac{\beta}{\alpha})^{n+1} - \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}}{2+2(\frac{\beta}{\alpha})^{n} - \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}} \ .\end{aligned}$$

Note that $|\beta/\alpha| < 1$, and thus we have

$$\lim_{n\to\infty} \frac{FM_{n+1}}{FM_n} = \alpha \lim_{n\to\infty} \frac{2+2(\frac{\beta}{\alpha})^{n+1} - \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}}{2+2(\frac{\beta}{\alpha})^{n} - \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}} = \alpha\,\frac{2-\frac{1}{\alpha-\beta}}{2-\frac{1}{\alpha-\beta}} = \alpha \ ,$$

and thus (5.5) follows.

Using the Equation (2.17), we can write

$$\frac{FM_{-(n+1)}}{FM_{-n}} = -\frac{FM_{n+1}+2F_{n+1}}{FM_n+2F_n} \ .$$

It follows from Binet's formula that

$$\begin{aligned}\frac{FM_{n+1}+2F_{n+1}}{FM_n+2F_n} &= \frac{2(\alpha^{n+1}+\beta^{n+1}) + \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}}{2(\alpha^n+\beta^n) + \frac{\alpha^n-\beta^n}{\alpha-\beta}} = \frac{\alpha^{n+1}\left[2+2(\frac{\beta}{\alpha})^{n+1} + \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}\right]}{\alpha^n\left[2+2(\frac{\beta}{\alpha})^{n} + \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}\right]} \\ &= \alpha\,\frac{2+2(\frac{\beta}{\alpha})^{n+1} + \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}}{2+2(\frac{\beta}{\alpha})^{n} + \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}} \ .\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left[\frac{FM_{n+1}+2F_{n+1}}{FM_n+2F_n}\right] = \alpha \lim_{n\to\infty} \frac{2+2(\frac{\beta}{\alpha})^{n+1} + \frac{1-(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}}{2+2(\frac{\beta}{\alpha})^{n} + \frac{1-(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}} = \alpha\,\frac{2+\frac{1}{\alpha-\beta}}{2+\frac{1}{\alpha-\beta}} = \alpha \ .$$

Hence, $\lim_{n\to\infty} \frac{FM_{-(n+1)}}{FM_{-n}} = -\alpha$. □

In what follows, we can immediately establish the next result using fundamental tools from the calculus of limits, along with (5.5) and (5.6).

Corollary 5.12. *If FM_n are the n-th term of the Fibonacci–Mulatu sequence, then* $\lim_{n\to\infty} \frac{FM_n}{FM_{n+1}} = \frac{1}{\alpha}$ *, and* $\lim_{n\to\infty} \frac{FM_{-n}}{FM_{-(n+1)}} = -\frac{1}{\alpha}$*, where α is the golden ratio* $(1+\sqrt{5})/2$.

6 Partial Sum Formulas

In this section, we present results on partial sums of terms of the Fibonacci–Mulatu numbers with n integers, and sum of the squares of the first n terms. Initially, consider the sequence of partial sums $S_n = FM_0 + FM_1 + FM_2 + \cdots + FM_n$, for $n \geq 0$, where $\{FM_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

A direct and immediate consequence of Lemma 2.5 is the result we now present. This naturally follows from the established relationships and supports the overall conclusions drawn from the proposition.

Proposition 6.1. *Let $\{FM_n\}_{n\geq 0}$ be the Fibonacci–Mulatu sequence. For all non-negative integers n, we have the following formulas:*

a) $\sum_{k=0}^{2n+1} (-1)^k FM_k = 7 - FM_{2n}$ *;*

and

b) $\sum_{k=0}^{2n+2} (-1)^k FM_k = 7 + FM_{2n+1}$.

Proof. a) First consider that n is odd, that is, the last term is negative, so

$$\begin{aligned}
\sum_{k=0}^{2n+1} (-1)^k FM_k &= FM_0 - FM_1 + FM_2 - FM_3 + \cdots + FM_{2n} - FM_{2n+1} \\
&= (FM_0 + FM_2 + \cdots + FM_{2n}) - (FM_1 + FM_3 + \cdots + FM_{2n+1}) \\
&= \sum_{k=0}^{n} FM_{2k} - \sum_{k=0}^{n} FM_{2k+1} .
\end{aligned}$$

According to the Lemma 2.5, items (b) and (c), it follows that: $\sum_{k=0}^{2n+1}(-1)^k FM_k = (FM_{2n+1} + 3) - (FM_{2n+2} - 4) = FM_{2n+1} - FM_{2n+2} + 7$. We obtain the result, using the Equation (1.1).

b) In which case that n is even, so

$$\begin{aligned}
\sum_{k=0}^{2(n+1)} (-1)^k FM_k &= FM_0 - FM_1 + FM_2 - FM_3 + \cdots + FM_{2n} - FM_{2n+1} + FM_{2n+2} \\
&= \sum_{k=0}^{n+1} FM_{2k} - \sum_{k=0}^{n} FM_{2k+1} = \sum_{k=0}^{n} FM_{2k} + FM_{2n+2} - \sum_{k=0}^{n} FM_{2k+1} \\
&= FM_{2n+1} + 7.
\end{aligned}$$

As in item (a), apply the Lemma 2.5. ☐

Now we describe the sum of the squares of the first n terms of the Fibonacci–Mulatu sequence in terms of the Fibonacci sequence.

Proposition 6.2. *Let $\{FM_n\}_{n\geq 0}$ be the Fibonacci–Mulatu sequence. The sum of the squares of the first n terms of the Fibonacci–Mulatu sequence is given by:*

$$FM_0^2 + FM_1^2 + FM_2^2 + FM_3^2 + \cdots + FM_{n-1}^2 + FM_n^2 = 12 + FM_n \cdot FM_{n+1} \ ,$$

for all non-negative integers n.

Proof. Let us first note that for $n \geq 2$, we have: $FM_nFM_{n+1} - FM_{n-1}FM_n = FM_n(FM_{n+1} - FM_{n-1}) = FM_n^2$. Thus, we find:

$$\begin{aligned} FM_2^2 &= FM_2FM_3 - FM_1FM_2, & FM_3^2 &= FM_3FM_4 - FM_2FM_3, \\ \ldots \quad FM_{n-1}^2 &= FM_{n-1}FM_n - FM_{n-2}FM_{n-1}, & FM_n^2 &= FM_nFM_{n+1} - FM_{n-1}FM_n. \end{aligned}$$

By summing both sides of these equalities, we obtain: $FM_2^2 + FM_3^2 + \cdots + FM_{n-1}^2 + FM_n^2 = FM_nFM_{n+1} - FM_1FM_2$. Since $FM_0 = 4$, $FM_1 = 1$ and $FM_2 = 5$ the result follows. ☐

7 Final Considerations

In this paper, we deal with three types of numbers: Fibonacci numbers, Fibonacci–Lucas numbers, and Fibonacci–Mulatu numbers. These may be more succinctly described as Fibonacci-type sequences, given that each element is calculated as the sum of the two preceding elements, provided that two initial terms are given. There is a substantial corpus of literature that examines each of these sequences in isolation. In this paper, we explore the connections between them. It appears that the results presented here are novel contributions to the existing literature. This work presents a comprehensive examination of the Fibonacci–Mulatu sequence, encompassing its intrinsic characteristics and a collection of classical identities. We have established connections with the Fibonacci and Fibonacci–Lucas sequences, which enabled us to derive significant results in the theory of Fibonacci–type sequences.

The fundamental relation involving the Fibonacci and Fibonacci–Lucas sequences was explored to derive a Binet-like formula for the Fibonacci–Mulatu sequence. Subsequently, we obtained exponential and Poisson-generating functions, which can be applied in future work to explore the combinatorial properties related to Fibonacci-type sequences. Moreover, several classical identities were examined, including those attributed to d'Ocagne, Catalan, Cassini, Melham, and Cesàro, as well as the convolution identity. However, as potential avenues for future research, we propose investigating the existence of specific and unique identities for this sequence. Furthermore, we present identities for the partial sums of the Fibonacci–Mulatu sequence terms, partial sums of alternating terms, and partial sums of the squares of the terms of this sequence. Furthermore, in future work, the Fibonacci–Mulatu numbers can be studied in another set of numbers and from other perspectives, such as matrices and combinatorial.

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Author contributions

Every author contributed equally to each part of the paper.

Conflict of interest statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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