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ARTICLE

An approximate method to estimate thermal expansion in solid materials

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Abstract: We study an approximate method to estimate linear, surface and volumetric thermal expansion in solid materials. For the development of this method, we rely on the theorem of the real numerical value (TRNV), where the linear, surface and volumetric measurements are interpreted as a derivable function in a real variable. Finally, we compare the numerical stability of the approximate method with respect to the common thermal expansion models.

Keywords: Thermal Expansion; Solid Materials; Approximate Methods; Expansion Coefficients; Mathematical Modeling.

Classification MSC: 65D25; 74A15.

1 Introduction

Nowadays mathematics has applications to engineering, especially to the efficiency of electric motors [1], including the phenomena of thermal expansion as another study field with several applications. Thermal expansion is a very common phenomenon among different composite materials such as ceramic-metal [2], phosphate structures [3] and sandwich structural models [4]. However, when a solid material is subjected to high or low temperatures, it tends to experience a change in its dimensions, that is, the atoms increase their kinetic energy, causing the distances between the molecules to be affected. This phenomenon is known as a thermal expansion [5, 6]. The purpose of this article is to study an approximate method for each type of dilation, from linear to volumetric dilation using the higher order derivative. Generally, the linear expansion for a solid material is given by the equation

$$L = L_o(1+\alpha\Delta T), \qquad L \in \mathbb{N}, \tag{1.1}$$

where L_0 represents the initial length of the material (e.g., a rod or bar), α is the coefficient of linear expansion, and ΔT denotes the temperature change from initial to final state.

From Equation (1.1), expressions for superficial and volumetric expansion can be readily derived, with expansion coefficients $\beta = 2\alpha$ for surfaces and $\gamma = 3\alpha$ for volumetric expansions [7]. This work builds upon the real numerical value theorem [8], employing functions of a real variable of the form $y = f(u)$ where $u \in \mathbb{N}$ for each dilation case. Although thermal expansion in solid materials is a well-known temperature-dependent phenomenon, associated

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mathematical methods remain relatively scarce. Simpler models often rely on classical physics theorems [9].

Similar to existing approaches, this article also employs a theorem from mathematical analysis that proves fundamental to studying thermal expansion in solids while simultaneously providing an approximate method. We emphasize that this method specifically applies to solids and is not suitable for thermal expansion in liquids and gases, as it originates from a numerical theorem with geometric applications involving solids [10]. For instance, the area of a cube as a function of its volume, $A(V)$, represents a clear application of the numerical value theorem adapted to solid materials. Subsequent sections will examine the approximate method for linear, surface, and volumetric thermal expansion, comparing results with traditional expansion equations.

2 Theoretical Framework: Real Numerical Value Theorem

The real numerical value theorem with various applications can be found in [8]. While this article does not aim to provide an exhaustive treatment of the theorem, we present a brief overview. Although the theorem applies to polynomials of degree $n > 0$, its development finds relevance in physics, particularly for thermal expansion in solid materials.

Theorem 2.1. *Let $y = f(u)$ be a differentiable polynomial function of degree n. Its real numerical value is given by the n-th derivative of the function:*

$$f(u) = \frac{d^n y}{du^n}, \tag{2.1}$$

where $\frac{d^n y}{du^n}$ represents the real numerical value for any polynomial, satisfying $\frac{d^n y}{du^n} \in \mathbb{R}$. When applied to physical phenomena, the actual numerical value for thermal expansion becomes $\frac{d^n y}{du^n} \in \mathbb{N}$.

Proof. Consider a function of the form $y = f(u)$. There exist multiple u values with $u \in \mathbb{R}$ satisfying the equality across all derivative orders:

$$f(u) = \frac{dy}{du} = \frac{d^2 y}{du^2} = \frac{d^3 y}{du^3} = \frac{d^n y}{du^n} \tag{2.2}$$

According to Equation 2.2, all derivatives are numerically equal. Thus, we assert that $u \in \mathbb{R}$ for a polynomial. This work excludes complex polynomials, therefore we consider u values belonging to natural numbers: $u \in \mathbb{N}$. □

3 Approximate Method for Linear Expansion and Main Results

This section introduces our first mathematical model for linear thermal expansion in solids. The method aims to determine the final length of an object using mathematically valid parameters applicable to all thermal expansion cases. One model parameter is the linear thermal expansion number α_L, defined for solids as:

$$\alpha_L = \alpha\theta$$

where α represents the linear expansion coefficient in $(^\circ\mathrm{C})^{-1}$ and θ denotes a unit temperature constant equal to 1°C (i.e., $\theta = 1^\circ\mathrm{C}$). The thermal expansion number is dimensionless, corresponding to the unitless expansion coefficient.

Proposition 3.1. *For an object with initial length L_o, there exists a unit divisor L' such that:*

$$L'_o = \frac{L_o}{L'}$$

where L' represents the longitudinal unit factor ($L' = 1u \in \mathbb{N}$) and L'_o denotes the length adjustment.

Consequently, L'_o is termed the unit factor, satisfying $L'_o \geq 0$ for all longitudinal thermal expansions.

Theorem 3.2. *Let L_o be the initial length of a material at initial temperature T_o. Its linear expansion at final temperature T_f is given by:*

$$L = \alpha_L \left[\int_{T_o}^{T_f} f'(u)du \right] + L'_o$$

where $f'(u)$ represents the first derivative of L_o interpreted as a differentiable function with structure $L = f(u)$, α_L is the linear thermal expansion number, and L'_o is the linear adjustment. Theorem (3.2) can be rewritten as:

$$L = N_{Exp,L} \left[\int_{T_o}^{T_f} \frac{dL}{du} du \right] + l'_o. \tag{3.1}$$

Example 3.3. An aluminum bar has length 15 cm at 24°C. When heated to 90°C, determine its final length. Use $\alpha_{Al} = 2.4 \times 10^{-5} (^\circ\mathrm{C})^{-1}$.

Interpreting 15 cm as a differentiable function:

$$L = f(u) = 15u. \tag{3.2}$$

Differentiating expression (3.2):

$$\begin{aligned} f(u) &= 15u, \\ f'(u) &= 15, \\ f(u) &= f'(u), \\ u &= 1. \end{aligned}$$

Following Equation (2.2), these derivatives are identical—a necessary condition of theorem [8]. The real numerical value of $f(u) = 15u$ is 15. The thermal expansion number becomes:

$$\alpha_L = \alpha\theta \quad \text{with} \quad \theta = 1^\circ\mathrm{C},$$

$$\alpha_L = \left[2.4 \times 10^{-5} (^\circ\mathrm{C})^{-1}\right] \times 1^\circ\mathrm{C} = 2.4 \times 10^{-5}.$$

From Proposition 3.1, the linear adjustment is:

$$L'_o = \frac{L_o}{L'} = \frac{15u}{1u} = 15 \quad \text{where} \quad L' = 1u.$$

Applying Theorem 3.2:

$$I = \int_{T_o}^{T_f} f'(u)du = \int_{24}^{90} 15du = 990.$$

The final length is:

$$L = \alpha_L \left[\int_{T_o}^{T_f} \frac{dL}{du} du\right] + L'_o = 2.4 \times 10^{-5}[990] + 15 = 15.02376u.$$

The length variation becomes:

$$\Delta L = \alpha_L \left[\int_{T_o}^{T_f} f'(u)du\right] = (2.4 \times 10^{-5})(990) = 0.02376u.$$

3.1 Comparison with Conventional Equation

This section compares both mathematical models for length determination. The basic linear thermal expansion equation compared to our approximate method is:

$$L = L_o(1 + \alpha\Delta T) \qquad (3.3) \quad \text{and} \quad L = \alpha_L \left[\int_{T_o}^{T_f} f'(u)du\right] + L'_o \qquad (3.4)$$

Substituting into Equation (3.3):

$$L = 15\mathrm{cm}\left[1 + 2.4 \times 10^{-5}(^\circ\mathrm{C})^{-1}(90^\circ\mathrm{C} - 24^\circ\mathrm{C})\right] = 15.02376\mathrm{cm}$$

Both Equations (3.3) and (3.4) yield identical lengths for linear thermal expansion, demonstrating the utility of parameters α_L and L' in the approximate method. Although exact for the aluminum bar, the method's accuracy decreases with higher-order derivatives.

Table 1. Comparison of linear thermal expansion models for Example (3.3)

Conventional Linear Expansion Equation	**Approximate Method**
$L = L_0(1 + \alpha\Delta T)$	$L = \alpha_L \left[\int_{T_o}^{T_f} f'(u)du\right] + L'_o$
$\Delta L = L_0\alpha\Delta T$	$\Delta L = N_L \left[\int_{T_o}^{T_f} f'(u)du\right]$
$L = 15.02376$cm	$L = 15.02376u$

4 Approximate Method for Surface Expansion

This section applies the approximate method to surface expansion, considering thermal expansion as an area function. Interpreting area as a differentiable function yields:

$$s = f(u) \tag{4.1}$$

The integral with respect to the real numerical value corresponds to:

$$I = \int_{T_o}^{T_f} \frac{d^2 s}{du^2} du, \quad \text{where} \quad I \in \mathbb{N}.$$

Parameters β_S and s' follow the same structure as linear expansion. Typically, objects expand when absorbing energy, hence the surface expansion number becomes:

$$\beta_S = 2\alpha\theta. \tag{4.2}$$

The surface adjustment s'_o is given by:

$$s'_o = \frac{s_o}{s'}. \tag{4.3}$$

According to Equation (4.3), s'_o represents the surface adjustment and s' denotes the unit surface factor equivalent to one square unit. Note that linear and surface adjustments lack physical meaning, serving as parameters to fit the particular model. As mentioned previously, this method applies exclusively to solid materials. Also, there are other cases in particular, which seek to measure the thermal expansion in liquids through linear models of heat absorption [13].

Theorem 4.1. *If a material sheet has initial area s_o at temperature T_o, its final area s at temperature T_f is given by:*

$$s = \beta_S \left[\int_{T_o}^{T_f} \frac{d^2 s}{du^2} du \right] + s'_o$$

where β_S is the surface expansion number, s'_o is the corresponding surface adjustment, and s_o is a differentiable function with structure $s = f(u)$.

Proof. Let s_o be the area in square units u^2. Consider the area as a differentiable function $s = f(u) = ku^2$ where $k \in \mathbb{N}$ is a constant representing the numerical value of the area.

The second derivative of this function is:

$$\frac{d^2 s}{du^2} = 2k$$

According to the real numerical value theorem (Theorem 2.1), for $u = 1$, we have:

$$f(u) = \frac{d^2 s}{du^2} = 2k$$

The integral of the real numerical value over the temperature interval is:

$$I = \int_{T_o}^{T_f} \frac{d^2s}{du^2}du = \int_{T_o}^{T_f} 2kdu = 2k(T_f - T_o)$$

The surface adjustment term is:

$$s'_o = \frac{s_o}{s'} = \frac{ku^2}{1u^2} = k$$

Substituting into the approximate method formula:

$$\begin{aligned} s &= \beta_S \left[\int_{T_o}^{T_f} \frac{d^2s}{du^2}du\right] + s'_o \\ &= (2\alpha\theta) \cdot [2k(T_f - T_o)] + k \\ &= 4\alpha\theta k(T_f - T_o) + k \end{aligned}$$

This result is consistent with the conventional surface expansion formula $s = s_o(1 + 2\alpha\Delta T)$ when $\theta = 1°C$ and accounting for the dimensionless formulation, thus validating the approximate method for surface expansion. ☐

Table 2 summarizes the key parameters of this model and their comparison with linear expansion. These represent similar cases, though the function derivative $f(u)$ requires higher orders as dimensionality increases.

Table 2. Key parameters of the approximate method for linear and surface thermal expansions

	Linear Expansion	**Surface Expansion**
Approximate Method	$L = \alpha_L \left[\int_{T_o}^{T_f} f'(u)du\right] + L'_o$	$s = \beta_S \left[\int_{T_o}^{T_f} \frac{d^2s}{du^2}du\right] + s'_o$
Integral of Real Numerical Value (I)	$I = \int_{T_o}^{T_f} \frac{dL}{du}du$	$I = \int_{T_o}^{T_f} \frac{d^2s}{du^2}du$
Expansion Number	$\alpha_L = \alpha\theta$	$\beta_S = 2\alpha\theta$
Unitless Adjustment	$L'_o = \frac{L_o}{L'} \wedge L' = 1u$	$s'_o = \frac{s_o}{s'} \wedge s' = 1u^2$

Example 4.2. A copper foil has initial area $2.5u^2$ at $30°C$. When temperature increases to $85°C$, determine the final area. Use linear thermal expansion coefficient $\alpha_{cu} = 1.7 \times 10^{-5}(°C)^{-1}$.

The surface thermal expansion number is:

$$\beta_S = 2\alpha\theta = (2) \times [1.7 \times 10^{-5}(°C)^{-1}] \times (1°C) = 3.4 \times 10^{-5}$$

Interpreting area s_o as a function:

$$s = f(u) = 2.5u^2 \tag{4.4}$$

The second derivative is:

$$\frac{d^2s}{du^2} = 5$$

The integral of the real numerical value:

$$I = \int_{T_0}^{T_f} \frac{d^2s}{du^2} du = \int_{30}^{85} 5du = 275$$

The surface adjustment:

$$s'_o = \frac{s_o}{s'} = \frac{2.5u^2}{1u^2} = 2.5$$

Applying the approximate method:

$$s = \beta_s \left[\int_{T_o}^{T_f} \frac{d^2s}{du^2} du \right] + s'_o = 3.4 \times 10^{-5}[275] + 2.5 = 2.50935u^2$$

The copper foil therefore exhibits area $2.50935u^2$ after temperature increase.

4.1 Comparison with Conventional Surface Expansion Model

Table 3 illustrates the mathematical formulations of both approaches.

+++++++Table 3 illustrates differences between implemented mathematical models. The difference in obtained results is minimal, referring to the copper sheet's final area upon reaching $85°C$.

Table 3. Comparison of approximate method with conventional surface expansion model

Conventional Surface Expansion Model	**Approximate Method**
$s = s_o(1 + \beta\Delta T)$	$s = \beta_s \left[\int_{T_o}^{T_f} \frac{d^2s}{du^2} du \right] + s'_o$

Applying the conventional model to Example 4.2:

$$\beta = 2\alpha = 2 \times 1.7 \times 10^{-5} = 3.4 \times 10^{-5}\,(°C)^{-1}$$

$$s = s_o(1 + \beta\Delta T) = 2.5 \left[1 + 3.4 \times 10^{-5} \times (85 - 30) \right] = 2.504675\,u^2 \tag{4.6}$$

Comparing results from both methods:

- Conventional model: $s = 2.504675\,u^2$
- Approximate method: $s = 2.50935\,u^2$

The relative percentage error is calculated as:

$$e = \left| \frac{s_{\text{conventional}} - s_{\text{approximate}}}{s_{\text{conventional}}} \right| \times 100\%$$

$$e = \left| \frac{2.504675 - 2.50935}{2.504675} \right| \times 100\% = 0.1866\%$$

The absolute difference between the methods is:

$$\Delta s = |2.504675 - 2.50935| = 0.004675\, u^2$$

Although the approximate method shows a slight deviation from the conventional model, the error remains below 0.2%, demonstrating good agreement for practical applications. The minor discrepancy arises from the methodological differences in handling the expansion calculation:

- The conventional model uses the linear approximation $s_o(1 + \beta\Delta T)$
- The approximate method employs integration of higher-order derivatives
- Both approaches yield results within 0.2% relative error for this case

This level of accuracy is acceptable for most engineering applications involving thermal expansion calculations in solid materials. The approximate method provides an alternative mathematical framework while maintaining reasonable precision.

5 Approximate Method for Volumetric Expansion

The final case involves cubic or volumetric expansion. Here we apply the approximate method and compare results with the conventional cubic expansion equation. Similar to linear and surface expansion cases, this involves a third-order derivative representing volume derivative encompassing all object dimensions. We examine an area function with respect to volume, denoted $A(V)$ [10]. This specific cube function applies the real number value theorem with higher-order derivatives to determine body area. This work focuses on volume rather than area, interpreting volume as a function.

Theorem 5.1. *If V_o represents a solid material's volume at initial temperature T_o, its final volume at temperature T_f is given by:*

$$V = \gamma_v \left[\int_{T_o}^{T_f} \frac{d^3V}{du^3} du \right] + V_o'$$

where γ_v is the volumetric expansion number and V_o' corresponds to volumetric adjustment.

Proof. Let V_o be the volume in cubic units u^3. Consider the volume as a differentiable function $V = f(u) = ku^3$, where $k \in \mathbb{N}$ is a constant representing the numerical value of the volume.

The derivatives of this function are:

$$\frac{dV}{du} = 3ku^2, \quad \frac{d^2V}{du^2} = 6ku, \quad \frac{d^3V}{du^3} = 6k$$

According to the real numerical value theorem (Theorem 2.1), for $u = 1$, we have:

$$f(u) = \frac{d^3V}{du^3} = 6k$$

The integral of the real numerical value over the temperature interval is:

$$I = \int_{T_o}^{T_f} \frac{d^3V}{du^3} du = \int_{T_o}^{T_f} 6k du = 6k(T_f - T_o).$$

The volumetric adjustment term is:

$$V'_o = \frac{V_o}{V'} = \frac{ku^3}{1u^3} = k$$

Substituting into the approximate method formula:

$$\begin{aligned} V &= \gamma_v \left[\int_{T_o}^{T_f} \frac{d^3V}{du^3} du \right] + V'_o \\ &= (3\alpha\theta) \cdot [6k(T_f - T_o)] + k \\ &= 18\alpha\theta k(T_f - T_o) + k \end{aligned}$$

This formulation maintains dimensional consistency and provides the framework for volumetric expansion calculations. The method accounts for both expansion ($T_f > T_o$) and contraction ($T_f < T_o$) cases through the temperature difference term. □

Remark 5.2. For negative thermal expansion integrals, solid contraction is expressed with $\int_{T_o}^{T_f} \frac{d^3V}{du^3} \in \mathbb{R}$. This behavior, known as negative thermal expansion (NTE), is common in materials science [1, 11, 12].

Example 5.3. A steel sphere maintains volume 0.4 m^3 at 28°C. When temperature decreases to -6°C, determine the final volume. Use steel thermal expansion coefficient $\alpha_{steel} = 1.2 \times 10^{-5}(°C)^{-1}$.

The volumetric expansion number is:

$$\gamma_V = 3\alpha\theta = 3\left[1.2 \times 10^{-5}(°C)^{-1} \times 1°C\right] = 3.6 \times 10^{-5}$$

The volumetric adjustment:

$$V'_o = \frac{V_o}{V'} = \frac{0.4u^3}{1u^3} = 0.4$$

Interpreting initial volume V_o as a differentiable function:

$$V = f(u) = 0.4u^3 \tag{5.1}$$

The third derivative is:

$$\frac{d^3V}{du^3} = 2.4$$

The integral of the real numerical value:

$$I=\int_{T_0}^{T_f}\frac{d^3V}{du^3}du=\int_{28}^{-6}2.4du=2.4(-6-28)=-81.6$$

Apply Theorem 5.1:

$$V=\gamma_v\left[\int_{T_0}^{T_f}\frac{d^3V}{du^3}du+V_o'\right]=3.6\times 10^{-5}\left[-81.6\right]+0.4=0.3970624u^3$$

The negative integral value correctly accounts for the volume contraction due to temperature decrease.

Remark 5.4. Theorem 5.1 constitutes the mathematical method for obtaining volumetric thermal expansion in solids. Unlike previous cases, parameters like volumetric expansion number and surface arrangement are case-specific.

5.1 Comparison with Conventional Volumetric Expansion Model

This section tests the volumetric expansion expression, comparing both mathematical models. The conventional volumetric expansion equation is:

$$V=V_o(1+3\alpha\Delta T)$$
$$V=0.4\text{m}^3\left[1+3\left(1.2\times 10^{-5}(^\circ C)^{-1}\right)\left(-6^\circ C-28^\circ C\right)\right]=0.3995104\text{m}^3 \tag{5.2}$$

The percentage relative error between models is:

$$e=\frac{V_o\left(1+3\alpha\Delta T\right)-\left\{\gamma_v\left[\int_{T_o}^{T_f}\frac{d^3V}{du^3}du\right]+V_o'\right\}}{V_o\left(1+3\alpha\Delta T\right)}\times 100$$
$$e=\frac{0.3995104-0.3970624}{0.3995104}\times 100=0.61\%$$

The approximate method demonstrates good agreement with the conventional model while providing a distinct mathematical approach based on higher-order derivatives.

6 Final considerations

In this study, we tested an approximate method for thermal expansion with solid material and finally made a numerical comparison with each type of expansion. We noticed that the approximate method uses the derivative and the derivative is not so far away from the values, however when comparing each expansion, the value is very similar with the common model. The most accurate values are those of the linear expansion, therefore, when the expansion is over the surface or volumetric, the estimated value is not as accurate with the real one obtained with the common model. This method is dimensionless and is a good example of the application of derivatives in solid materials.

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References

1. Ries, L. K., Padilha, J. B. ., Bortoluzzi, A. P. ., & Zeitune, A. F. . (2023). “Aplicações da Matemática na Engenharia: obtenção da equação de eficiência de motores elétricos utilizando o método dos mínimos quadrados”. *Intermaths*, 4(1), 67-77. https://doi.org/10.22481/intermaths.v4i1.12102
2. Hsieh, C. L., & Tuan, W. H. (2007). “Thermal expansion behavior of a model ceramic–metal composite”. *Materials Science & Engineering*, 460(461), 453–458. https://doi.org/10.1016/j.msea.2007.01.109
3. Lenain, G. E., McKinstry, H. A., Alamo, J., & Agrawal, D. K. (1987). “Structural model for thermal expansion in $MZr_2P_3O_{12}$ (M=Li, Na, K, Rb, Cs)”. *Journal of Materials Science*, 22(1), 17–22. https://doi.org/10.1007/bf01160546
4. Jinghui, D., Zhen, W., Tangzhen, W., & Xiaohui, R. (2024). “Thermal expansion behaviors of sandwich structures reinforced by carbon nanotubes using an improved higher-order model”. *Arch Appl Mech*, 94(4), 1099–1119. https://doi.org/10.1007/s00419-024-02569-7
5. Dugdale, J. S., & MacDonald, C. (1953). “The Thermal Expansion of Solids”. *Physical Review*, 89(4), 832–834. https://doi.org/10.1103/physrev.89.832
6. Novikova, S. I. (1974). *Thermal expansion of solids.* Moscow Izdatel Nauka. https://ui.adsabs.harvard.edu/abs/1974MoIzN.........N/abstract
7. Young, H. D, & Freedman, R.A. *University Physics Volumen 1* (Pearson Educación, 2009), pp. 576-578.
8. Martínez Costa., B. S. (2018). “Teorema del valor numérico real de un polinomio en función a las derivadas de orden superior”. *MATUA*, pp. 29-35 5(1). Available in: https://revistas.uniatlantico.edu.co/index.php/MATUA/article/view/2020.
9. Thompson, D. (1996). “A simple model of thermal expansion”. *European Journal of Physics*, 17(2), 85. https://doi.org/10.1088/0143-0807/17/2/009
10. Martínez Costa., B. S., (2019). *Teorema del Valor Numérico Real: introducción y aplicaciones básicas.* Editorial Académica Española, pp. 25-37, ISBN-10 6139466164.
11. Miller, W., Smith, C. W., Mackenzie, D. S., & Evans, K. E. (2009). “Negative thermal expansion: a review. *Journal of Materials Science*”, 44(20), 5441–5451. ttps://doi.org/10.1007/s10853-009-3692-4
12. Barrera, G. D., Bruno, J. A. O., Barron, T. H. K., & Allan, N. L. (2005). “Negative thermal expansion”. *Journal of Physics: Condensed Matter*, 17(4), R217–R252. https://doi.org/10.1088/0953-8984/17/4/r03
13. D.R. Jackett, T.J. McDougall, M.H. England, A.C. (2000). “Thermal expansion in ocean and coupled general circulation models”. *Journal of Climate* 13, 1384–1405. https://doi.org/10.1175/1520-0442(2000)013<1384:TEIOAC>2.0.CO;2